a place of mind

FACULTY OF EDUCATION

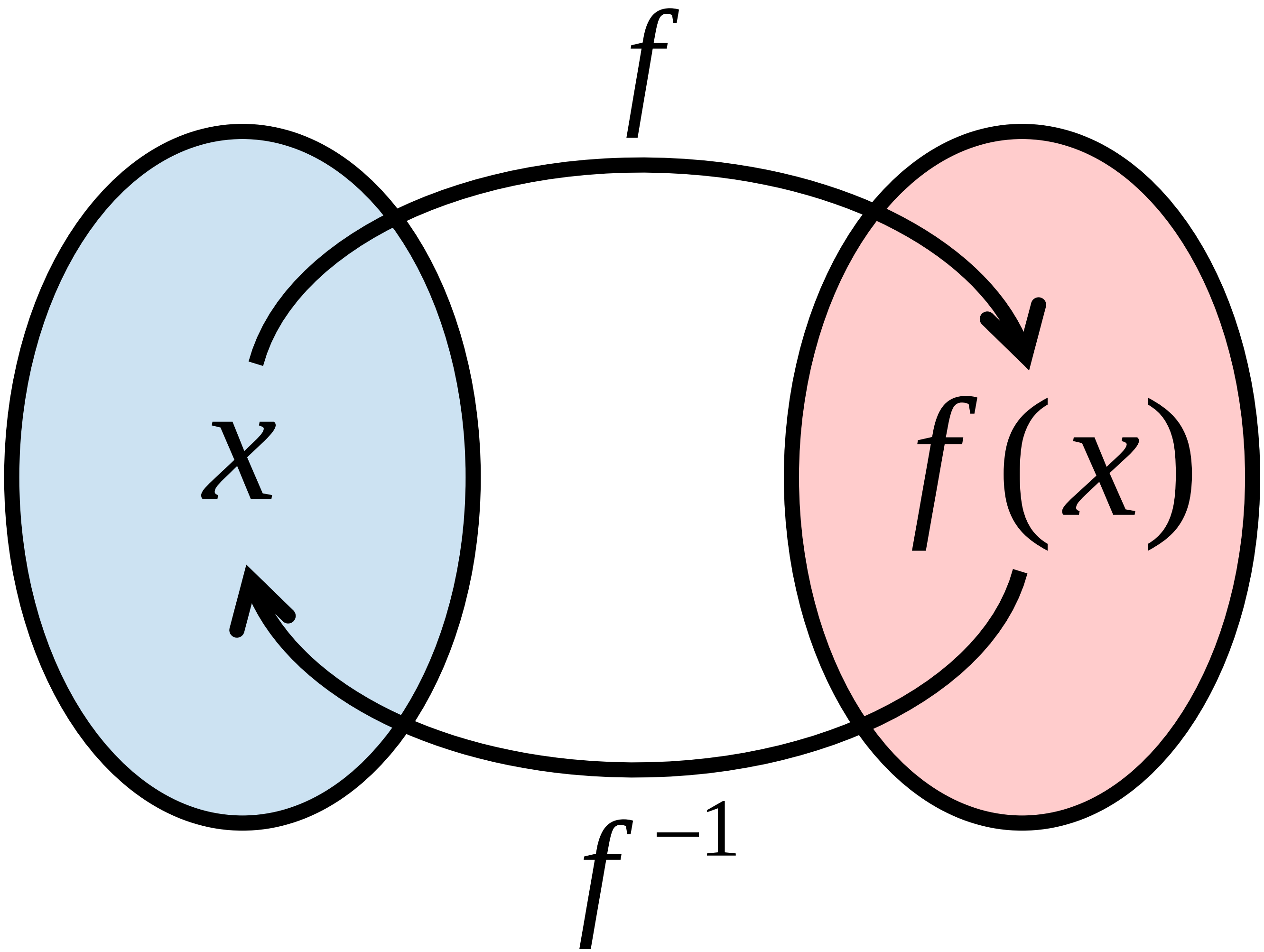
Department of
Curriculum and Pedagogy

Mathematics
Inverse Functions

Science and Mathematics
Education Research Group

Supported by UBC Teaching and Learning Enhancement Fund 2012-2014

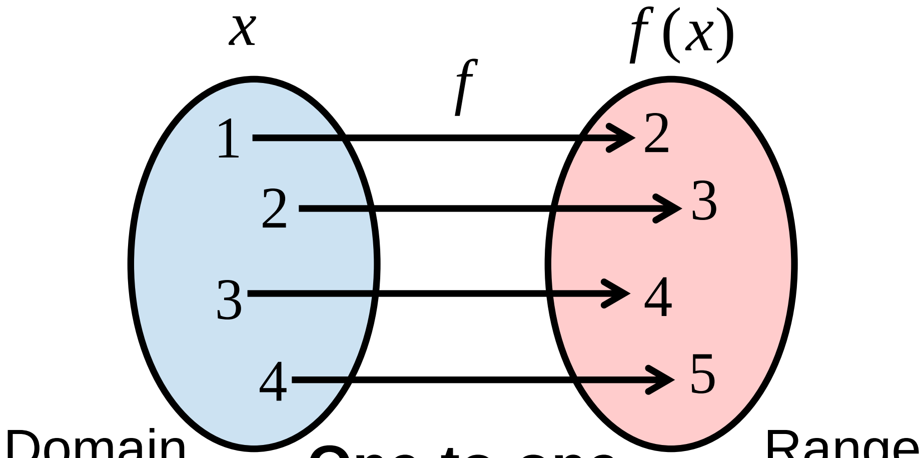
Inverse Functions



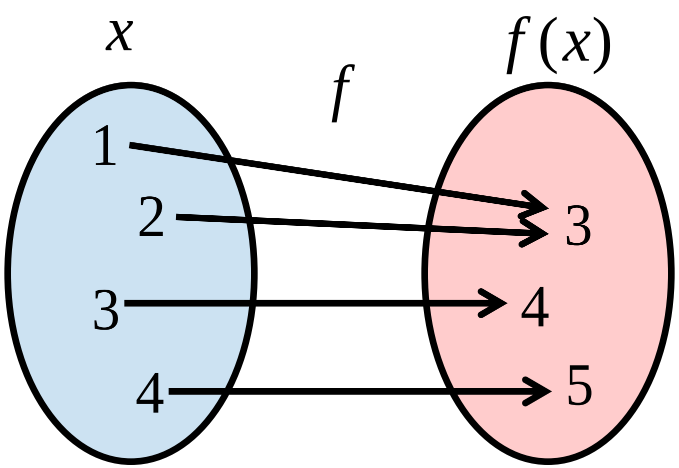
Definition: One-to-one

A function f is *one-to-one* if it does not map two different values in its domain to the same value in its range.

Knowing whether a function is one-to-one will become important when trying to find inverse functions. *While you may not need to know the definition of one-to-one,* you need to know how to determine if the inverse of a function is also a function.



One-to-one

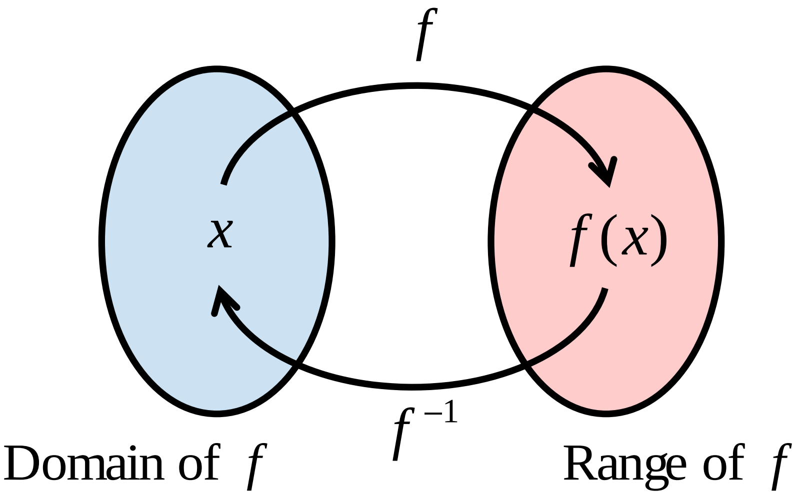


Not one-to-one

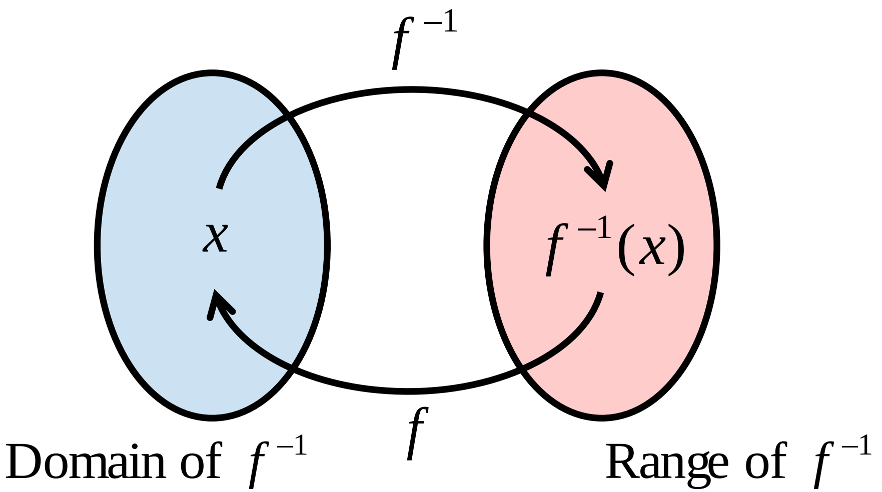
Definition: Inverse Function

Suppose the function f maps x to $f(x)$ as shown.

The inverse of f, denoted f^{-1}, undoes the mapping of f.



If the point (a, b) belongs to f, then the point (b, a) must belong to f^{-1}.



A function and its inverse have the property that:

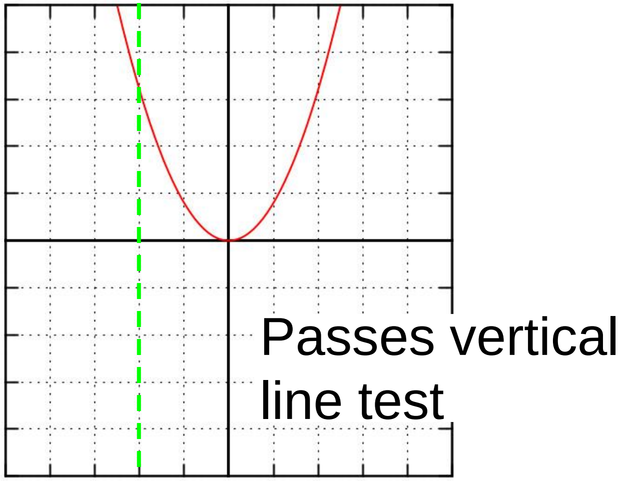
$$f^{-1}\left(f(x)\right)=x \quad \text{and} \quad f\left(f^{-1}(x)\right)=x$$

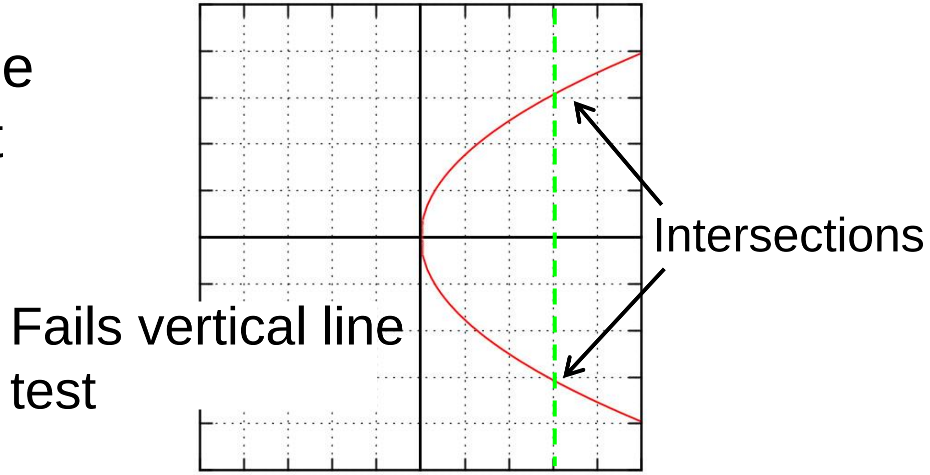
Definition: Vertical Line Test

Make sue you are familiar with functions before trying to learn about inverse functions.

Review: In order to test if a given graph represents a function, we can use the vertical line test on the graph.

Imagine drawing a vertical line through the graph. If this vertical line intersects the graph at two different points, then the graph does not represent a function.





Inverse Functions I

How many of the following are one-to-one functions?

A. 0

B. 1

C. 2

D. 3

E. 4

x	A(x)
1	1
2	2
3	3
4	4
5	5

x	B(x)
1	3
2	3
3	3
4	3
5	3

x	C(x)
3	1
3	2
3	3
3	4
3	5

x	D(x)
1	1
2	1
3	3
4	3
5	4

Solution

Answer: B

Justification: Function A always maps x to a different value. Since $A(x)$ is unique for all x, the function is one-to-one.

Function B always maps x to the same value, 3, so it is not one-to-one.

"Function C" is not a function because $x = 3$ maps to 5 different values. Therefore, it cannot be a one-to-one function.

Function D is not one-to-one because both $x = 3$ and $x = 4$ map to 3.

Inverse Functions II

How many of the following are one-to-one functions?

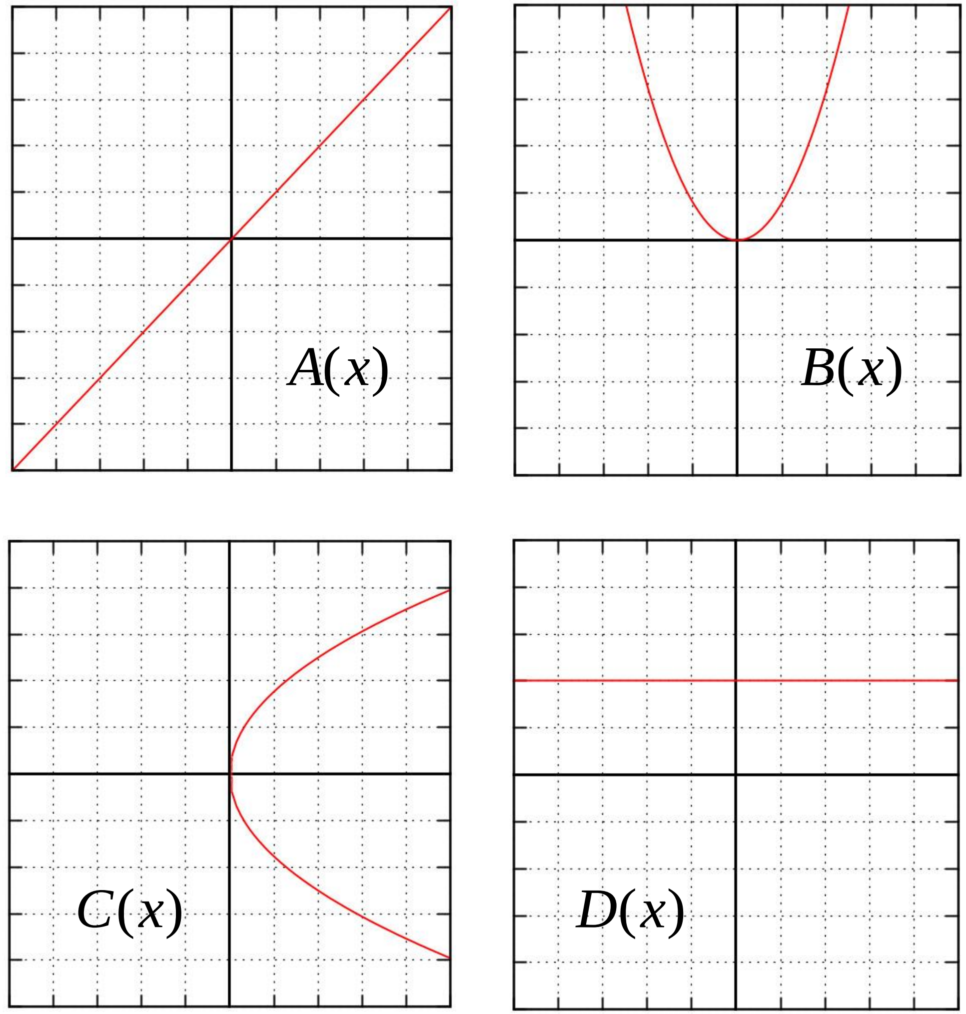
A. 0

B. 1

C. 2

D. 3

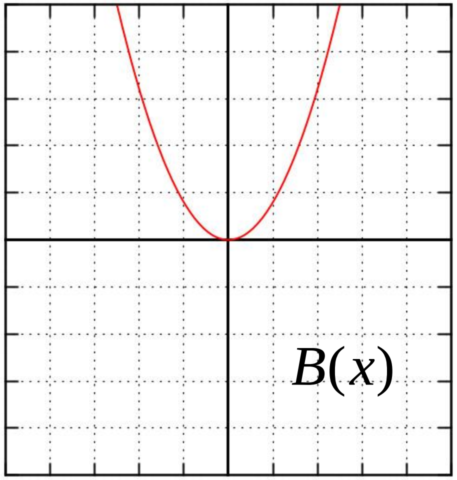
E. 4

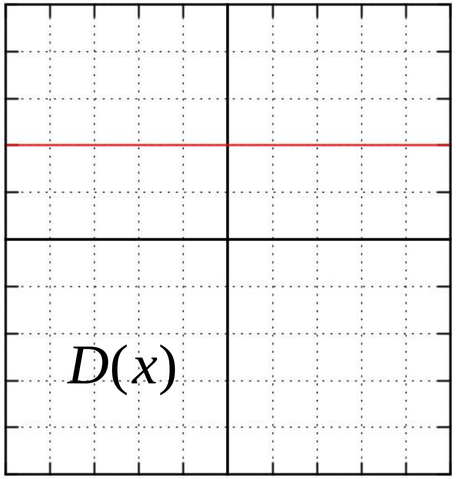


Solution

Answer: B

Justification: A function is not one-to-one if 2 different values of x map to the same value. This means that if 2 points lie on the same horizontal line, the function is not one-to-one:





Functions B and D fail the horizontal line test. This means that at least two different points lie on the same horizontal line. These functions are not one-to-one

Graph C is not a function since one value of x maps to 2 different values. This graph fails the vertical line test.

Inverse Functions III

Suppose the points $(a, f(a))$ and $(b, f(b))$ belong to a one-to-one function.

If $f(a) = f(b)$, what can be concluded about the relationship between a and b?

A. $a = b$

B. $a \neq b$

C. $a = b = 0$

D. $f(a) = f(b)$ is not possible for one-to-one functions

E. Nothing can be concluded about a and b

Solution

Answer: A

Justification: Recall that for one-to-one functions, no two x in the domain can map to the same value in the range. If $f(a) = f(b)$ this means that both $x = a$ and $x = b$ map to the same value.

In order for this to be possible, we must conclude that $a = b$, otherwise the function f is not one-to-one. The points $(a, f(a))$ and $(b, f(b))$ are the same point.

This fact is used to show that equations are one-to-one. If we assume that $f(a) = f(b)$ for two values a and b in the domain of f and show that $a = b$, then the function is one-to-one.

Extend Your Learning: Examples

Assume that $f(a) = f(b)$ for a and b in the domain of f.

Example 1: $f(x) = 2x + 1$

$$f(a) = f(b)$$
$$2a + 1 = 2b + 1$$
$$2a = 2b$$
$$a = b$$

Since it was shown that $a = b$, the function is one-to-one.

Example 2: $f(x) = x^2$

$$f(a) = f(b)$$
$$a^2 = b^2$$
$$a = \pm b$$

Since it may not be the case that $a = b$, the function is not one-to-one.

Inverse Functions IV

How many of the following are one-to-one functions?

$$A(x) = -2x + 1$$

$$B(x) = 3x^2 - 2x + 1$$

$$C(x) = x^3 - 1$$

$$D(x) = 2|x| + 1$$

A. 0
B. 1
C. 2
D. 3
E. 4

Solution

Answer: C

Justification: See the previous examples to show that function A is one-to-one and function B is not one to one.

Determine if function C and D are one-to-one:

$$C(x) = x^3 - 1$$
$$C(a) = C(b)$$
$$a^3 - 1 = b^3 - 1$$
$$a^3 = b^3$$
$$a = b$$

Function C is one-to-one

$$D(x) = 2|x| + 1$$
$$D(a) = D(b)$$
$$2|a| + 1 = 2|b| + 1$$
$$|a| = |b|$$
$$a = \pm b$$

Function D is not one-to-one

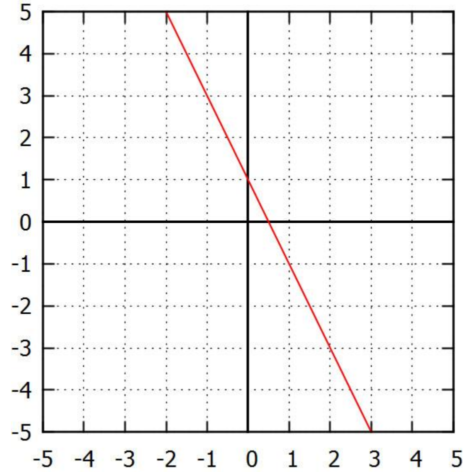
Alternative Solution

Answer: C

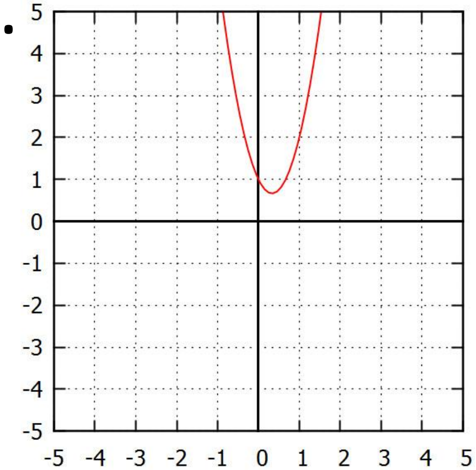
Justification: Alternatively you can graph each function and use the horizontal line test. This approach is often faster if you know the general shape of each function.

Functions A and C pass the horizontal line test, so they are one-to-one.

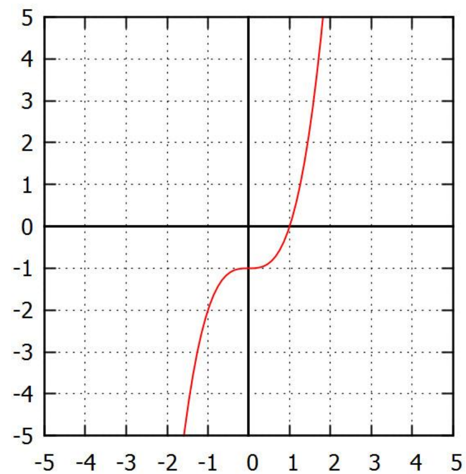
A.



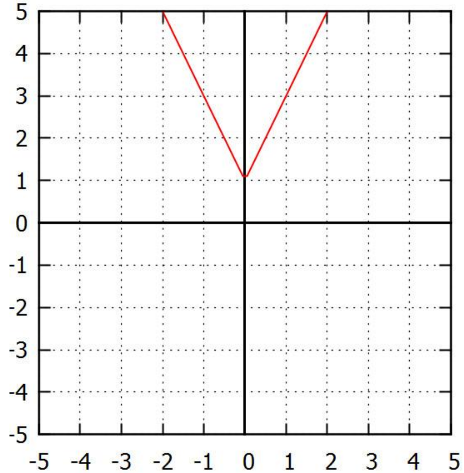
B.



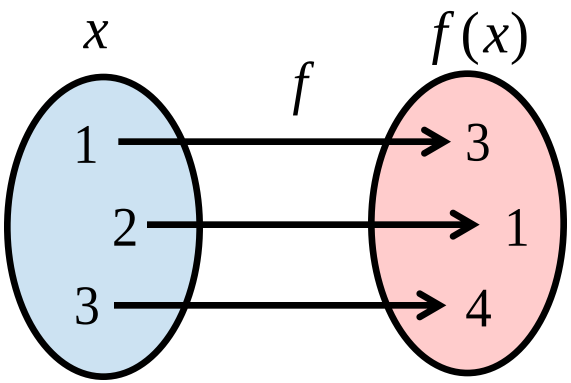
C.



D.

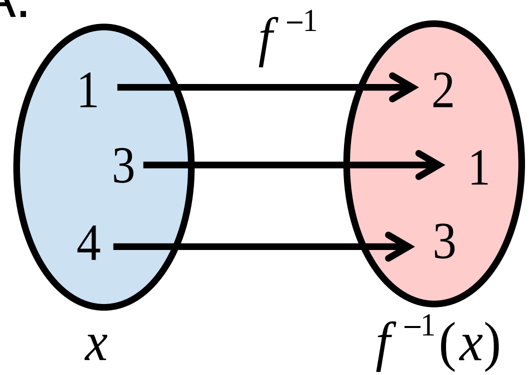


Inverse Functions V

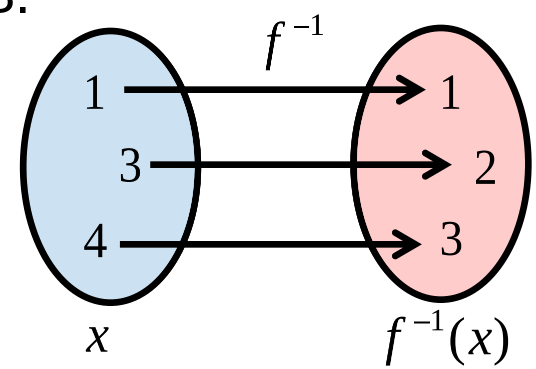


Which of the following is the correct inverse function of f ?

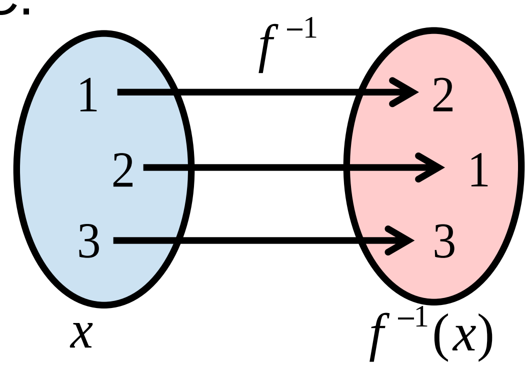
A.



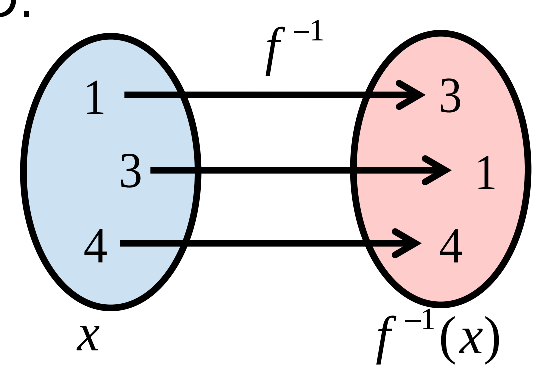
B.



C.



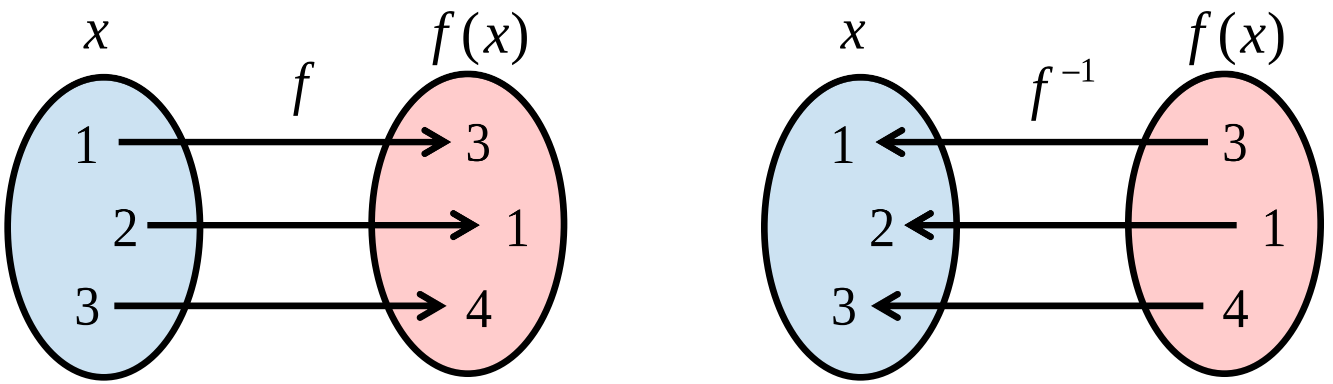
D.



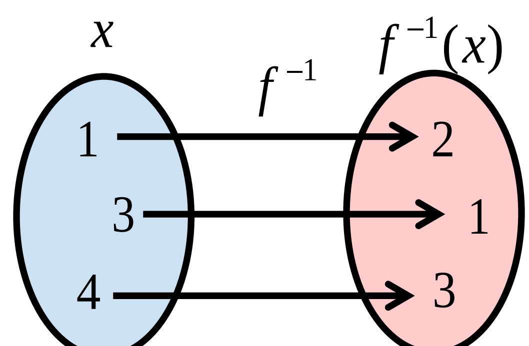
Solution

Answer: A

Justification: The inverse function f^{-1} maps values of $f(x)$ back to x.



Rearranging this as a function of x gives:



Inverse Functions VI

If the point (a, a+1) lies on the function f, which point lies on f^{-1}?

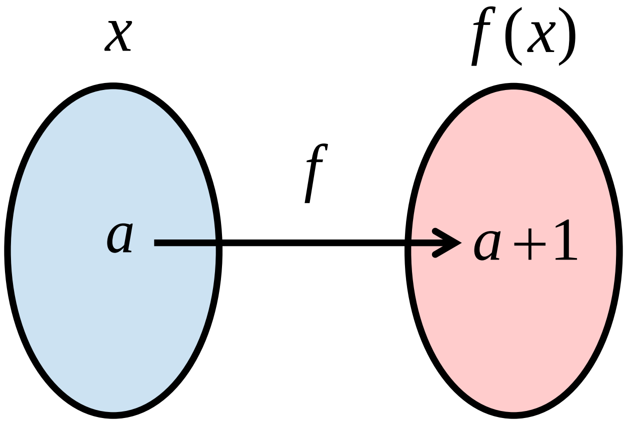
A. $(a, a-1)$

B. $(a-1, a)$

C. $(a, a+1)$

D. $(a+1, a)$

E. None of the above

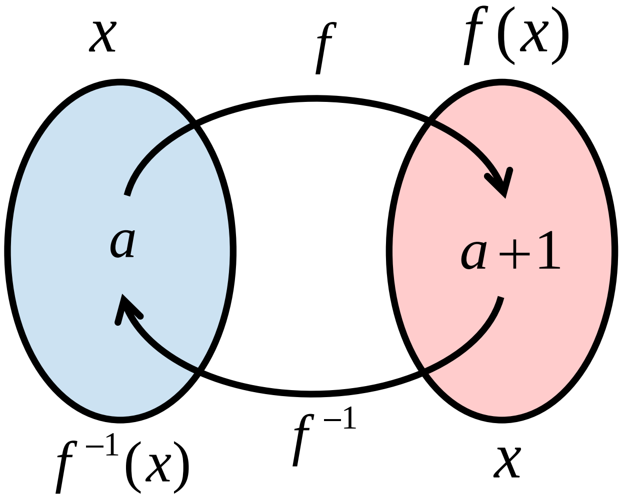


Solution

Answer: D

Justification: If the point (a, a+1) lies on function f, this means that $x = a$ was mapped to $f(a) = a + 1$.

The inverse function must then map $x = a + 1$ to $f^{-1}(a+1) = a$. The point (a+1, a) lies on f^{-1}.



Notice how the *x* and *y* coordinates of a function are interchanged in its inverse function.

Inverse Functions VII

The domain of function f is $x > 0$ and its range is all real numbers. What is the domain and range of f^{-1}?

	Domain	Range
A.	$x > 0$	All reals
B.	All reals	$y > 0$
C.	$x > 0$	$y > 0$
D.	All reals	All reals
E.	$x < 0$	All reals

Press for hint

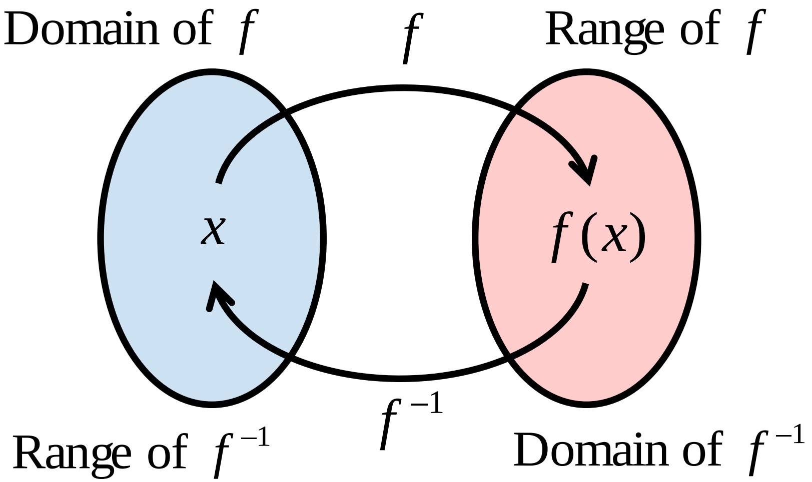
Consider the functions $f(x) = \ln(x)$ and its inverse $f^{-1}(x) = e^x$.

The domain of $f(x) = \ln(x)$ is $x > 0$, while its range is all real numbers.

Solution

Answer: B

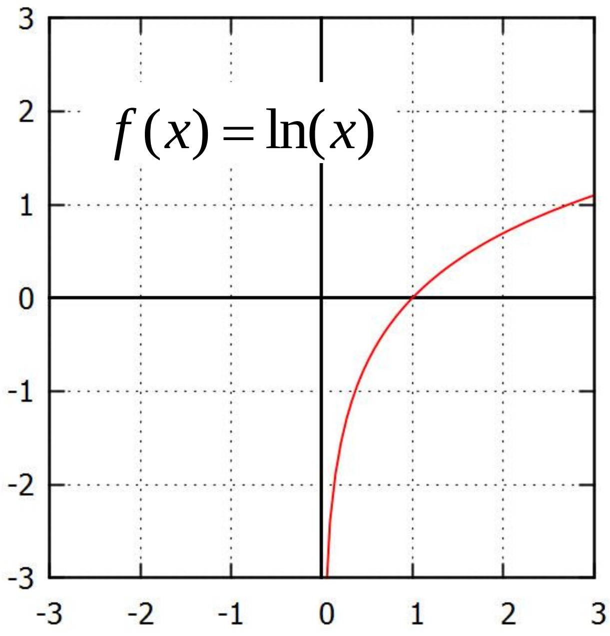
Justification: Recall the x and y values for each point of a function and its inverse interchange. This means the domain and range of a function and its inverse interchange.

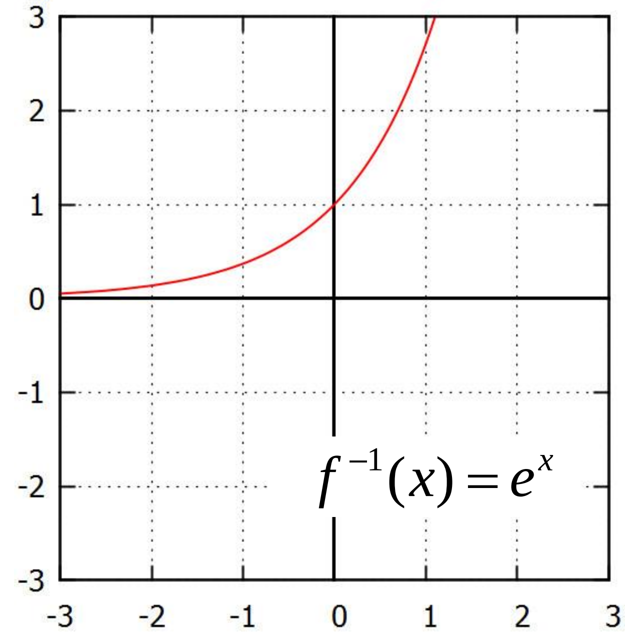


Since the domain of f is $x > 0$, the range of f^{-1} is $y > 0$. Likewise, the range of f is all real numbers, so the domain of f^{-1} is all real numbers.

Solution cont'd

Consider the graphs of $f(x) = \ln(x)$ and $f^{-1}(x) = e^x$. The domain and range of these two functions are the same as those asked in this question.





Domain: $x > 0$

Range: *All real numbers*

Domain: *All real numbers*

Range: $y > 0$

Inverse Functions VIII

Suppose that function f multiplies a number by 2, then adds one to it. Which of the following correctly describes its inverse?

A. Subtract 1, then divide by 2
B. Divide by 2, then subtract 1
C. Add 1, then divide by 2
D. Divide by 2, then add 1
E. Either A or B

Solution

Answer: A

Justification: The inverse function must undo the effects of the original function. Suppose x is put into function f. The value x will then be mapped to $f(x) = 2x + 1$. This function has the property that it will map $f^{-1}(x)$ back to x:

$$f(f^{-1}(x)) = x$$

$$2\left(f^{-1}(x)\right) + 1 = x \quad \text{since } f(x) = 2x + 1$$

$$f^{-1}(x) = \frac{1}{2}(x - 1) \quad \text{solve for } f^{-1}(x)$$

Following order of operations, we must first subtract one from the input then divide by 2.

Solution Continued

Answer: A

Justification: A quick short cut to finding the inverse of a function is letting $y = f(x)$, interchanging x and y, and then solving for y as a function of x.

$$f(x) = 2x + 1$$

$$y = 2x + 1 \quad \text{let } y = f(x)$$

$$x = 2y + 1 \quad \text{interchange } x \text{ and } y$$

$$2y = x - 1 \quad \text{solve for } y$$

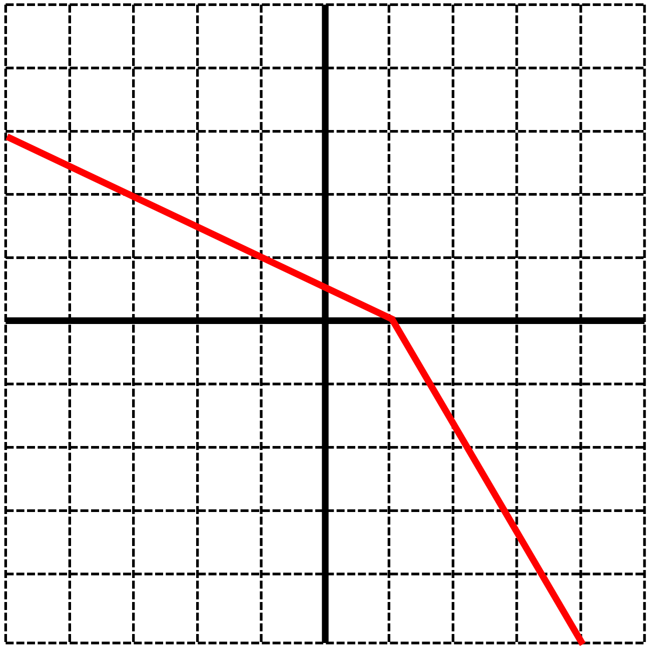
$$y = \frac{1}{2}(x - 1)$$

$$f^{-1}(x) = \frac{1}{2}(x - 1)$$

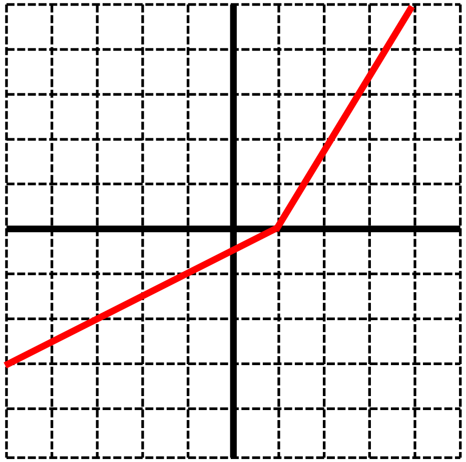
Compare this solution with the previous one to see why we can interchange the x and y.

Inverse Functions IX

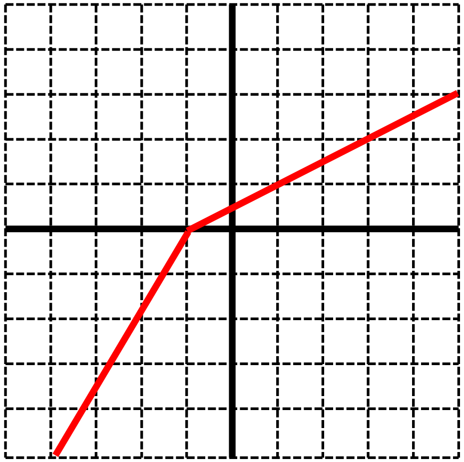
What is the inverse of the following function shown below?



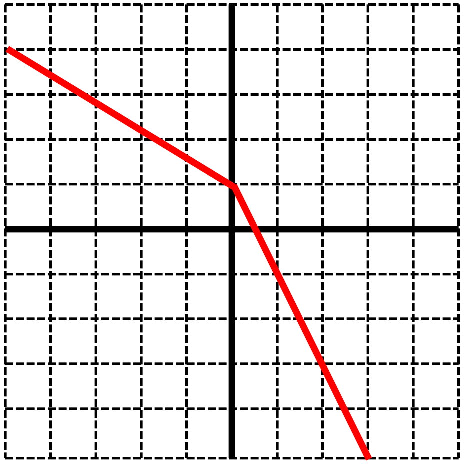
A.



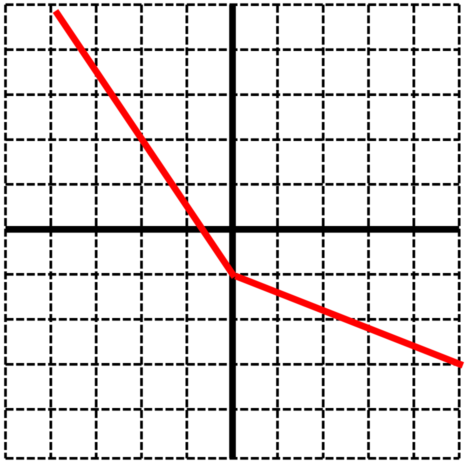
B.



C.



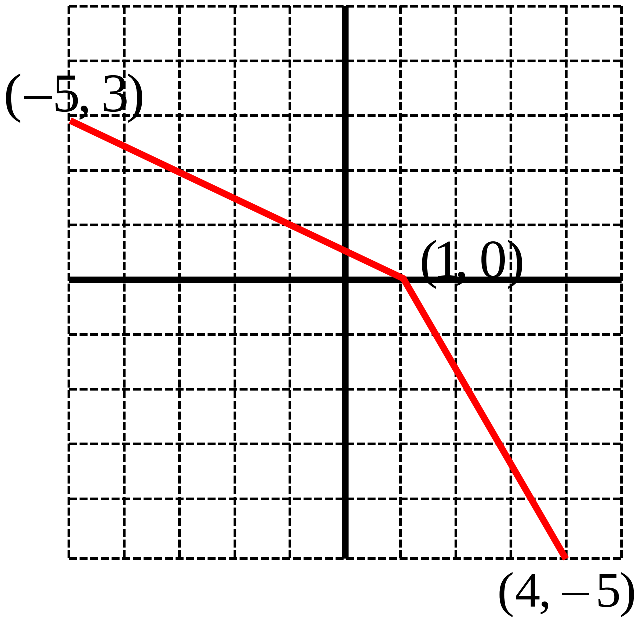
D.



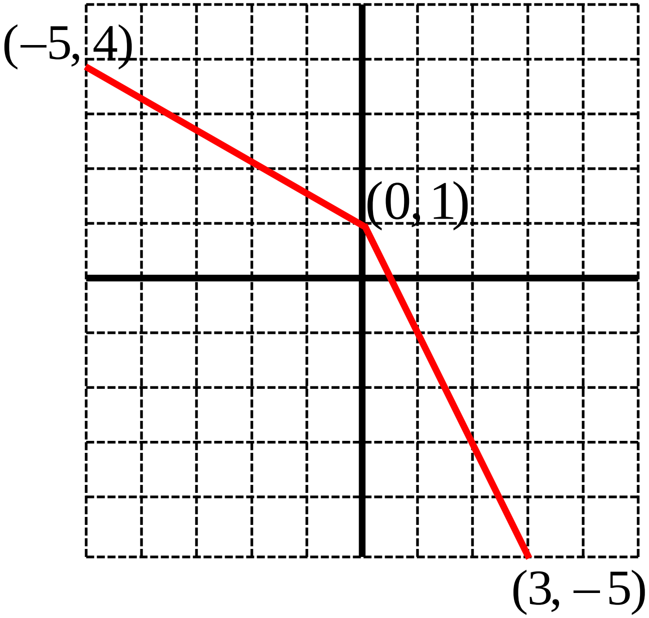
Solution

Answer: C

Justification: Find points on the original function, then interchange the x and y coordinates.



$$(-5, 3) \to (3, -5)$$
$$(1, 0) \to (0, 1)$$
$$(4, -5) \to (-5, 4)$$

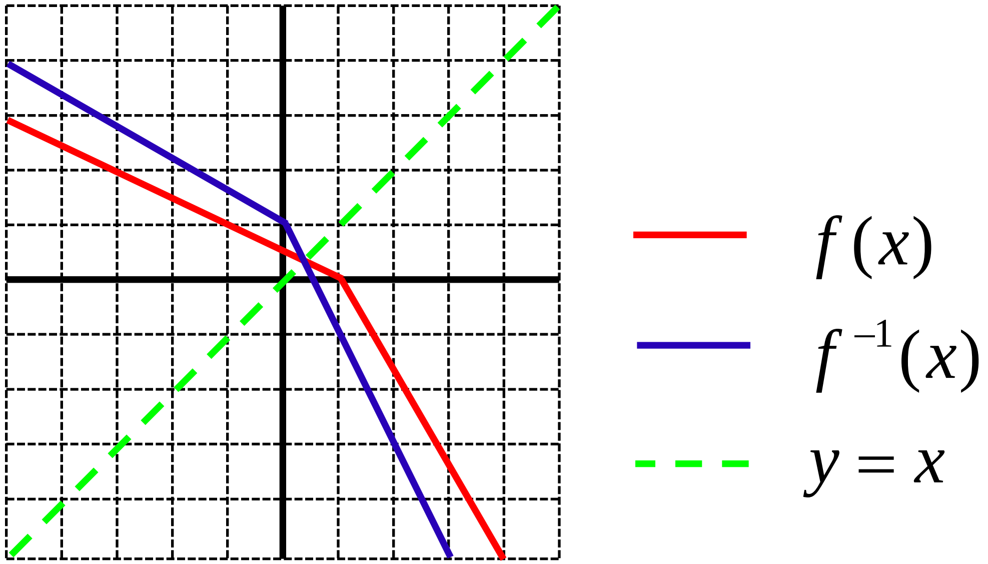


Solution Cont'd

Answer: C

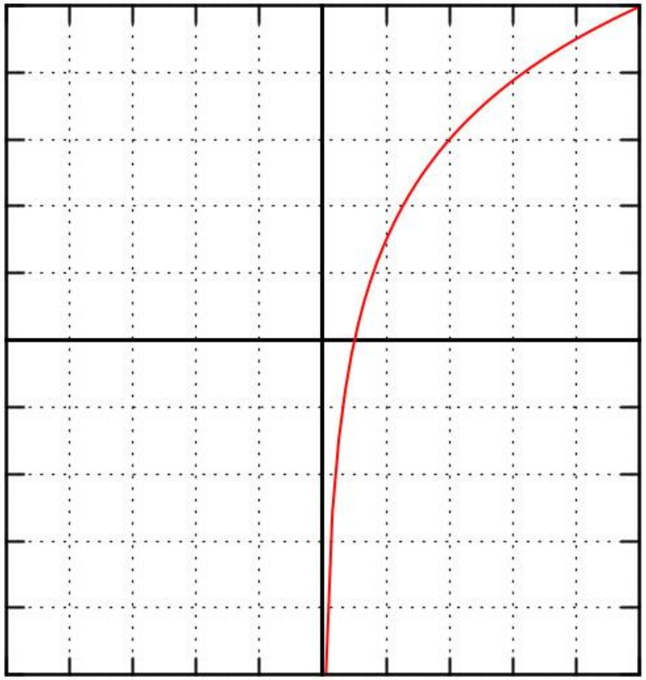
Justification: The inverse of a function is the reflection across the line $y = x$ because all x and y values interchange.

Answer A reflected the graph in the x-axis, answer B reflected the graph in the y-axis, and answer D reflected the graph in both the x and y axis.

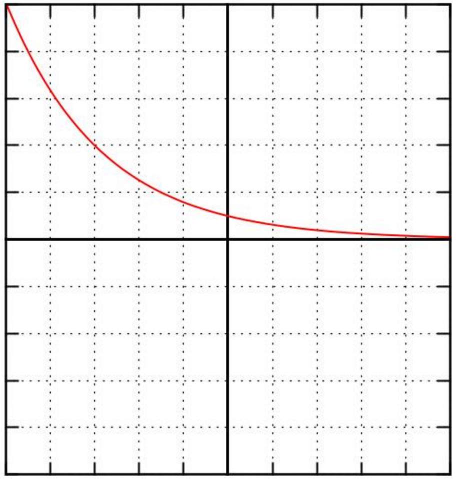


Inverse Functions X

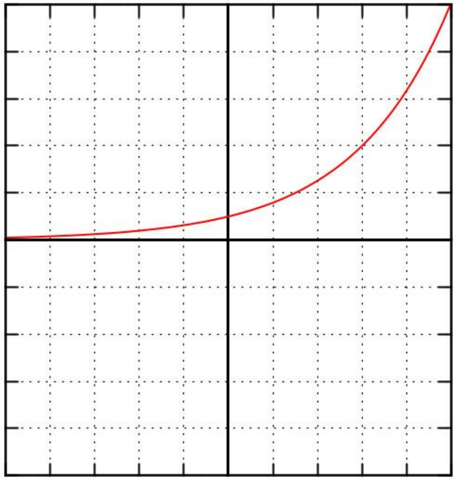
What is the inverse of the following function shown below?



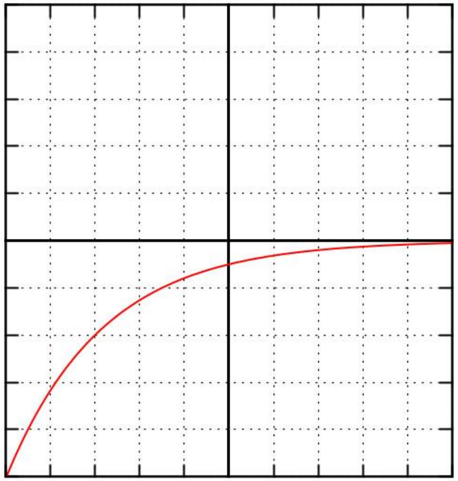
A.



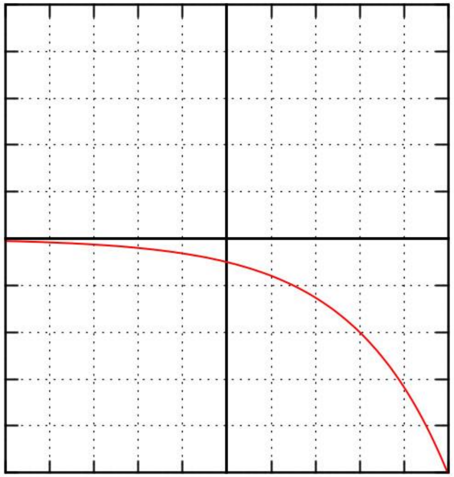
B.



C.



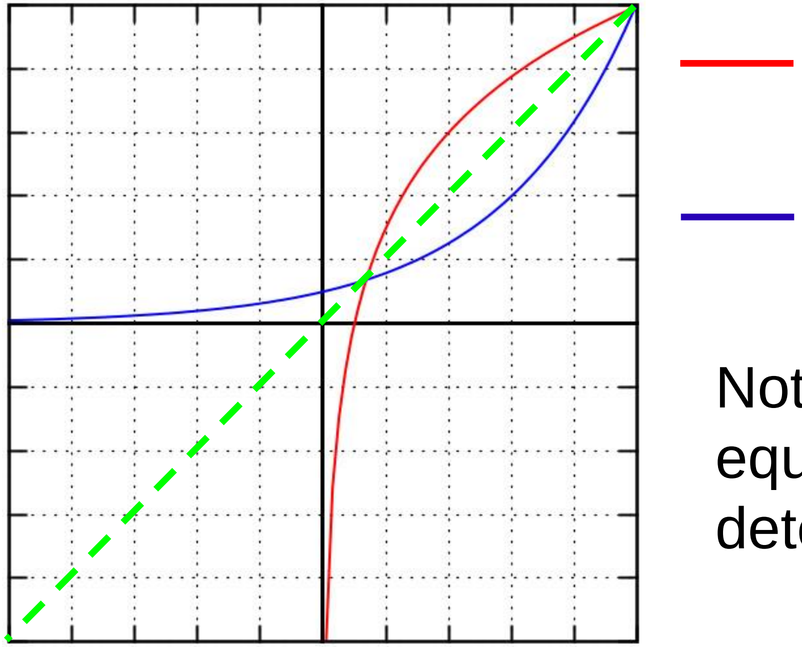
D.



Solution

Answer: B

Justification: Reflect the graph across the line $y = x$ to find the inverse.



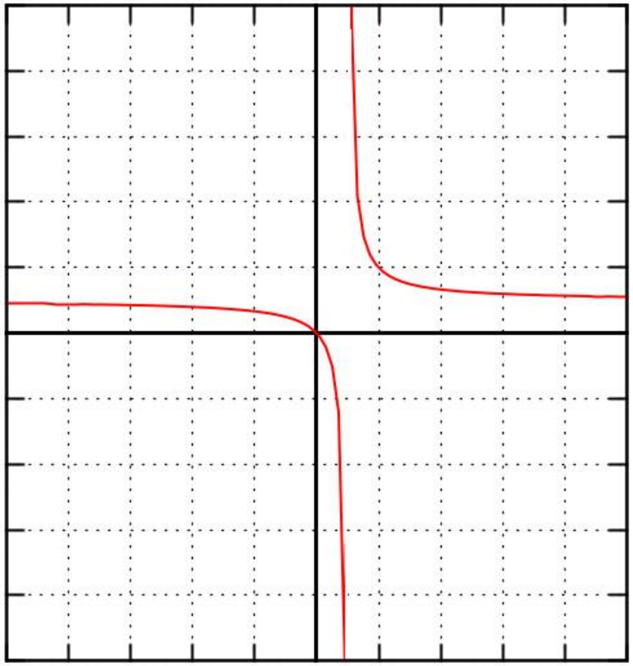
$$f(x) = 3\log_2 x$$

$$f^{-1}(x) = 2^{\frac{x}{3}}$$

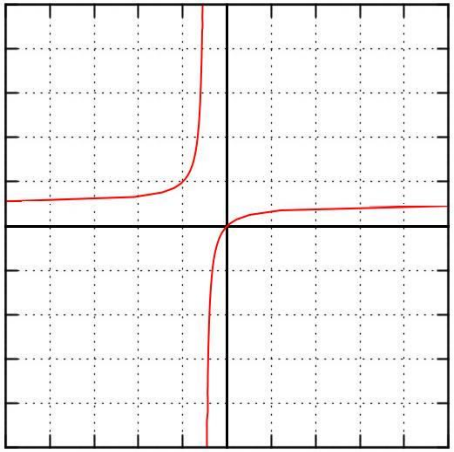
Note: It is not necessary to know the equations of the graphs shown to determine the inverse of a graph.

Inverse Functions XI

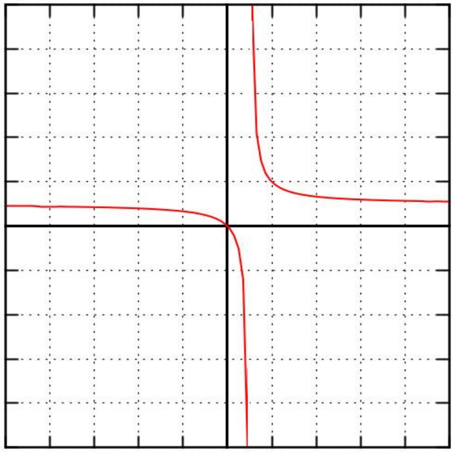
What is the inverse of the following function shown below?



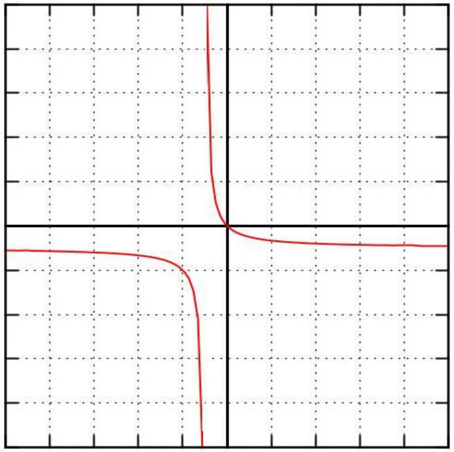
A.



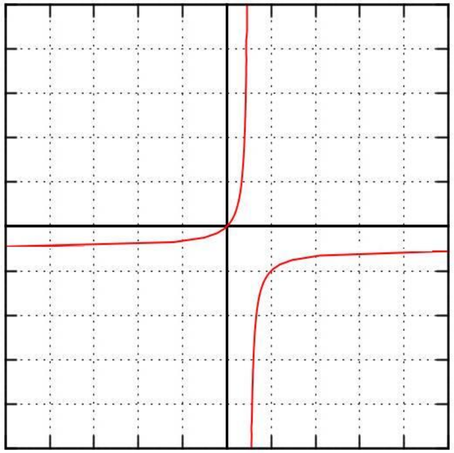
B.



C.



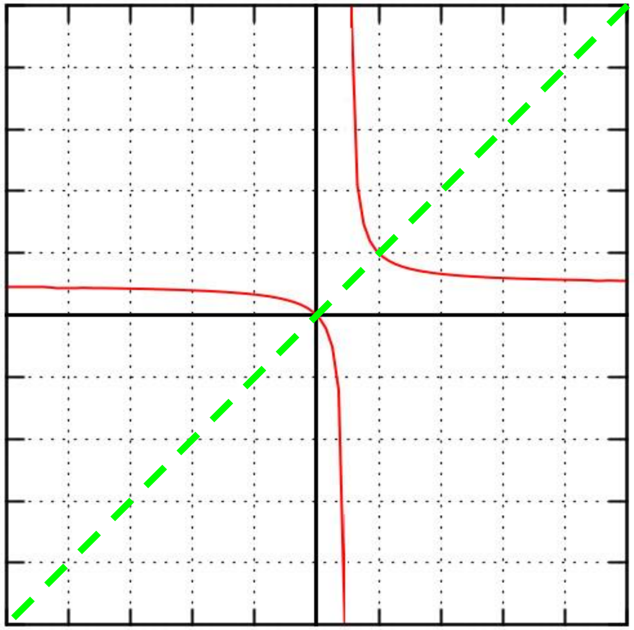
D.



Solution

Answer: B

Justification: This function is symmetric across the line $y = x$ and so a reflection across the line $y = x$ does not change the function. This function and its inverse are the same.



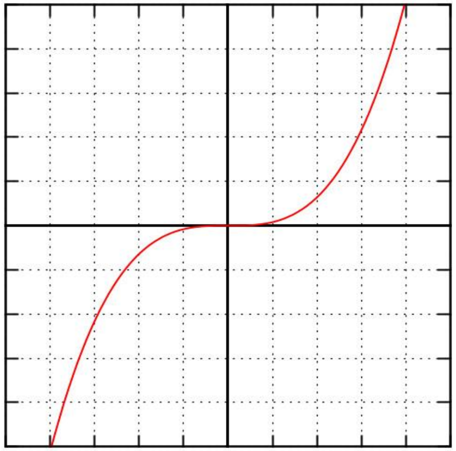
$$f(x) = \frac{1}{x-1} + 1$$

$$f^{-1}(x) = \frac{1}{x-1} + 1$$

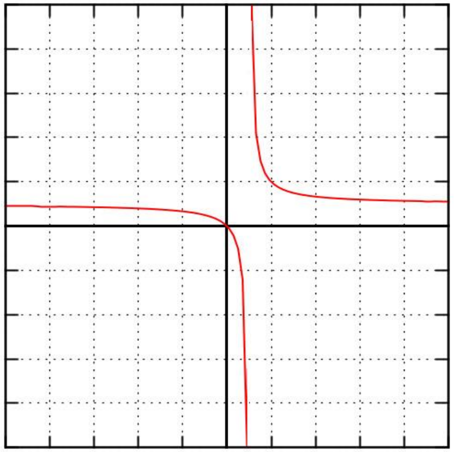
Inverse Functions XII

Of the four functions shown, which has an inverse that is not a function?

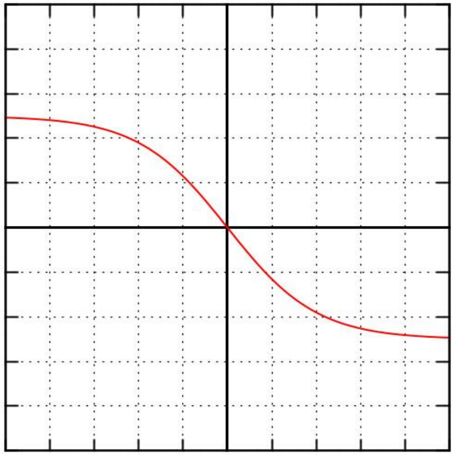
A.



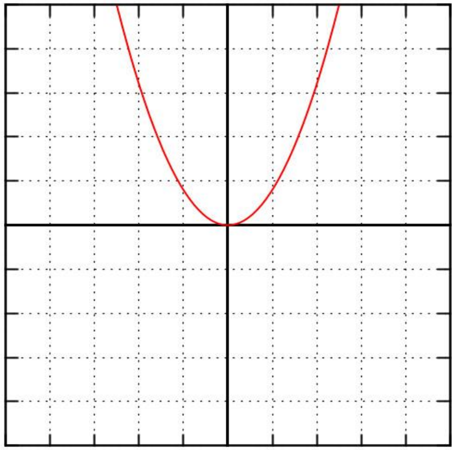
B.



C.



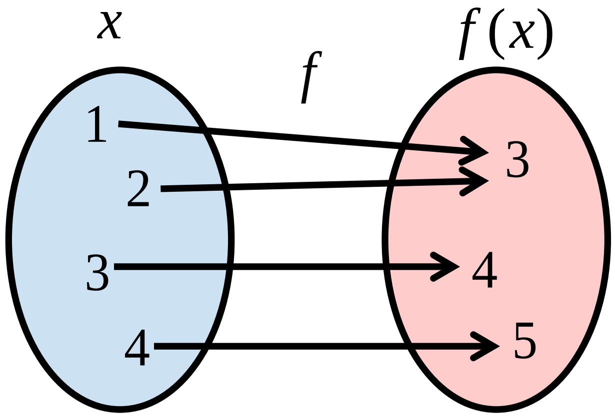
D.



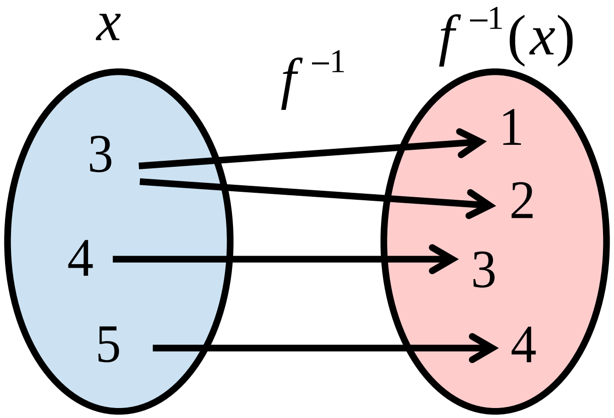
Solution

Answer: D

Justification: Only functions that are one-to-one have an inverse that is also a function.



Not one-to-one



Not a function

Since function D fails the horizontal line test (see question 2), it is not one-to-one and therefore its inverse is not a function.

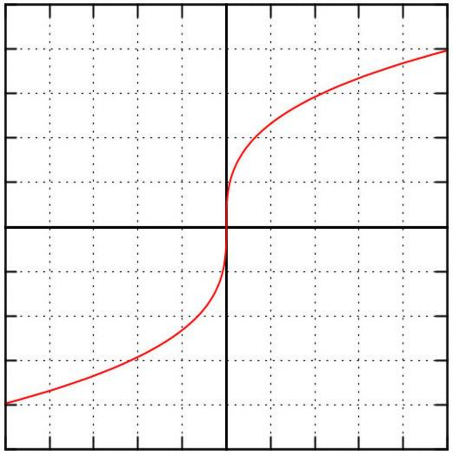
Solution Continued

The 4 graphs shown are the inverses of those in this question.

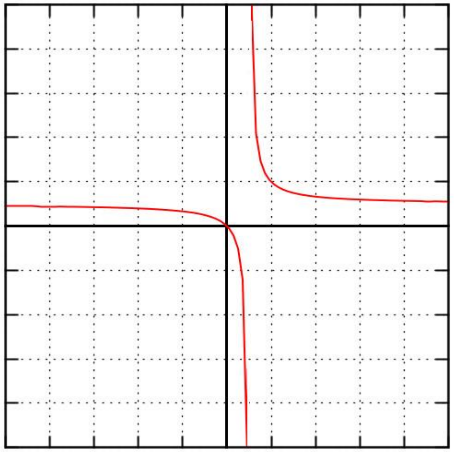
The inverse of graph D is the only graph that fails the vertical line test, so it is not a function.

To determine if the inverse of a function is a function. use the horizontal line test on the original function, or the vertical line test on the inverse.

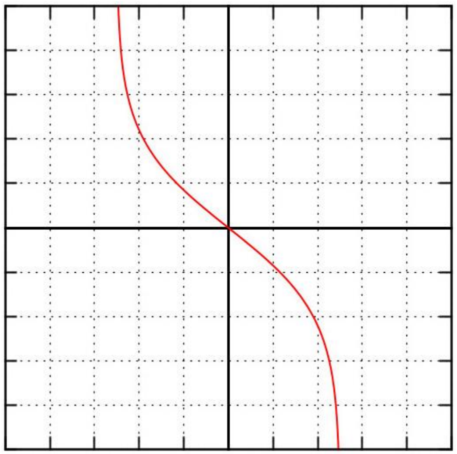
A.



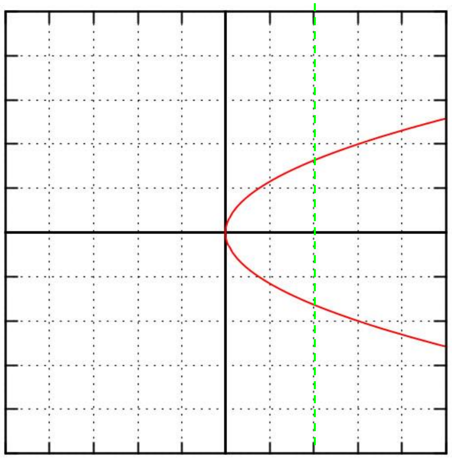
B.



C.



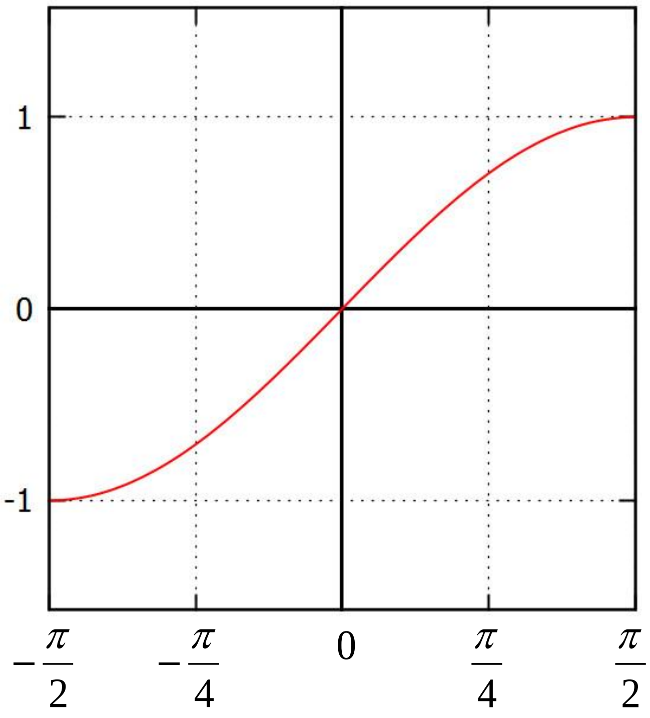
D.



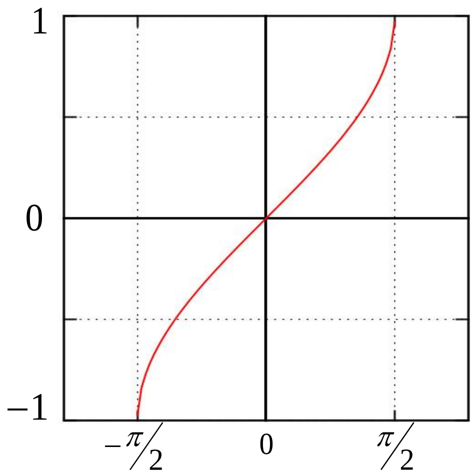
Inverse Functions XIII

Consider $f(x) = \sin(x)$ only for $-\pi/2 \leq x \leq \pi/2$. What is the inverse of f ?

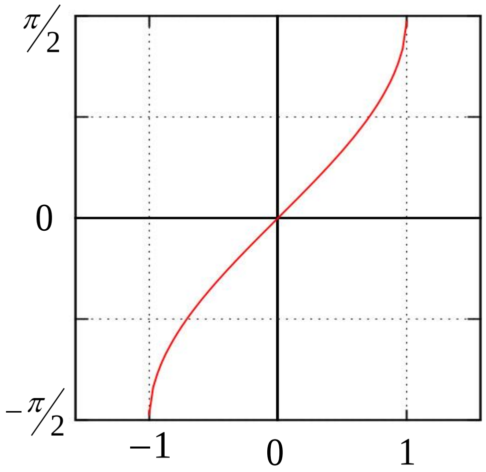
Note the different axes.



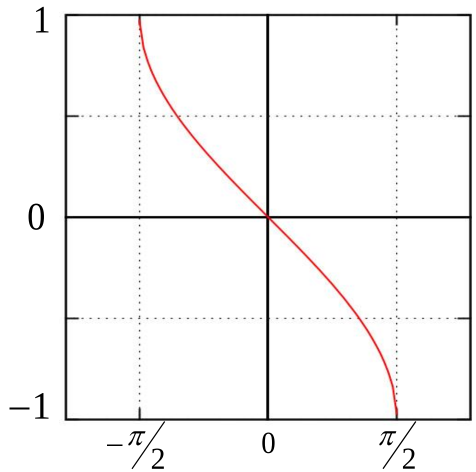
A.



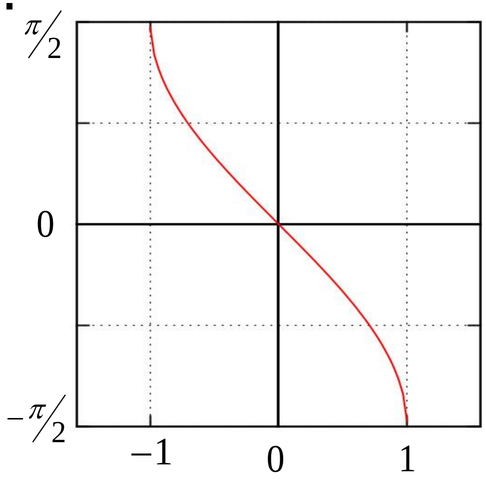
B.



C.



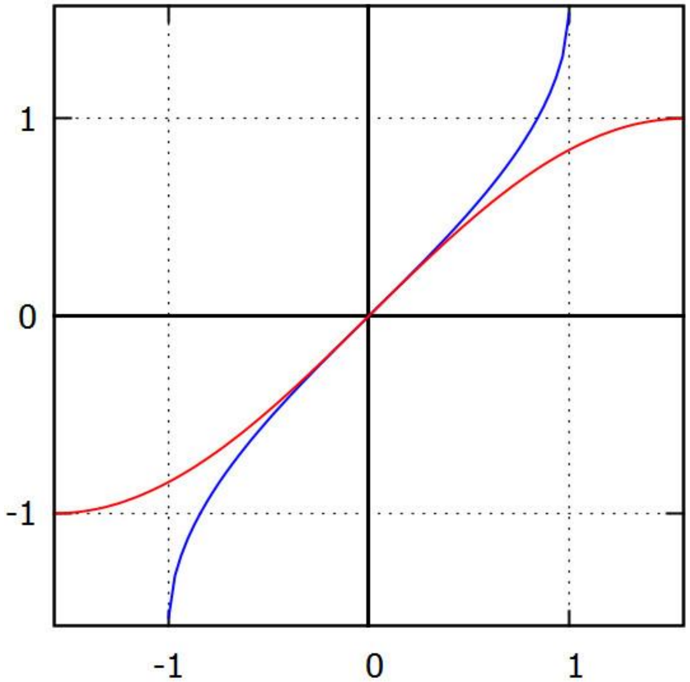
D.



Solution

Answer: B

Justification: Both answers A and B have the correct shape for f^{-1}.



$f(x) = \sin(x)$

$f^{-1}(x) = \sin^{-1}(x)$

Answers A and B differ by the labels on the x-axis and y-axis. Since the domain of f is $-\pi/2 \le x \le \pi/2$ and its range is $-1 \le y \le 1$, the domain of f^{-1} must be $-1 \le x \le 1$ and the range is $-\pi/2 \le y \le \pi/2$.

Note: The domain of $f(x) = \sin(x)$ is restricted so that f^{-1} is a function. The sine function is normally not one-to-one.