a place of mind

FACULTY OF EDUCATION

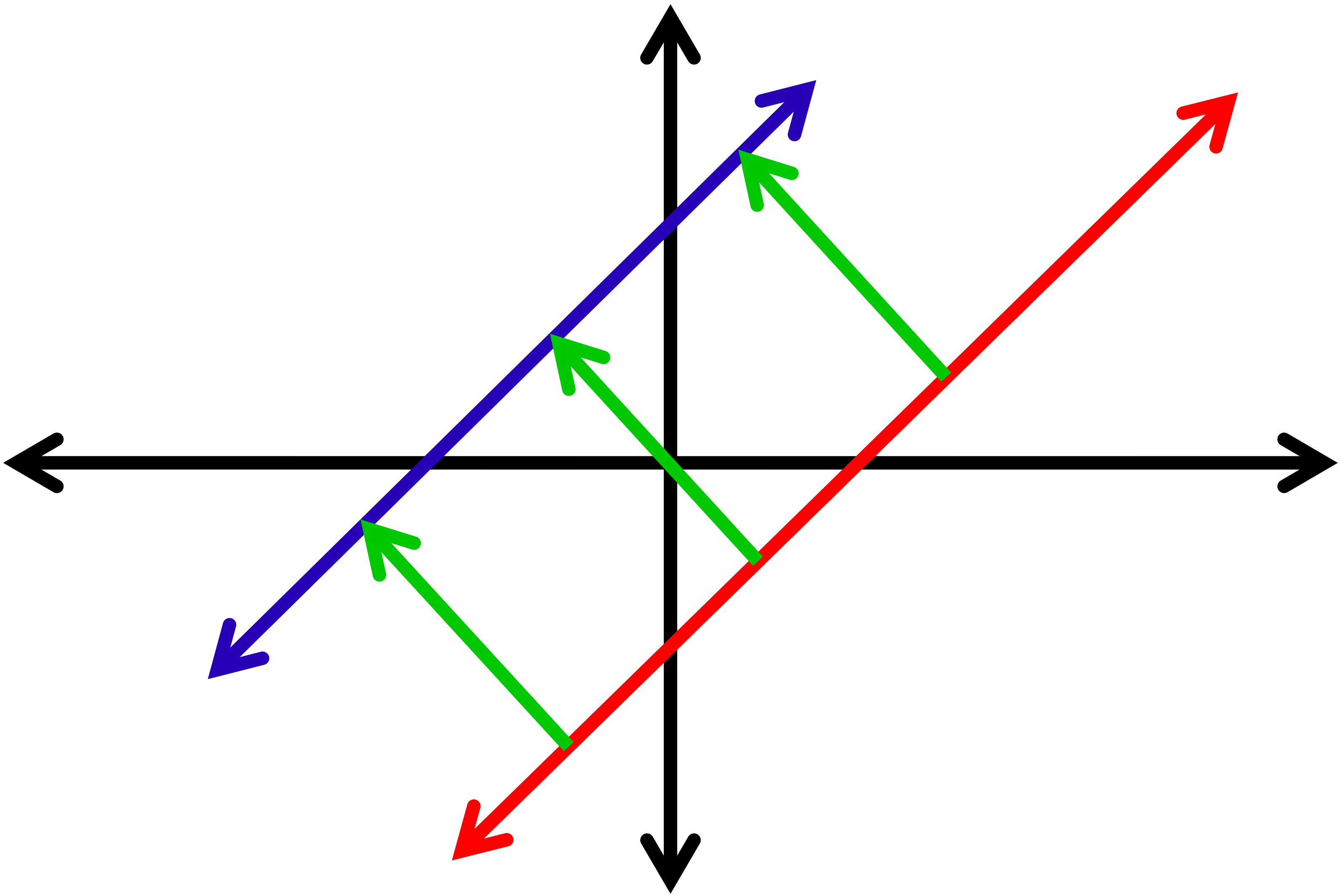
Department of
Curriculum and Pedagogy

Mathematics
Transformation of Lines

Science and Mathematics
Education Research Group

Supported by UBC Teaching and Learning Enhancement Fund 2012-2014

Transformation of Lines



Transformation of Lines I

Which one of the following is the correct value of $g(3)$ expressed in terms of $f(3)$?

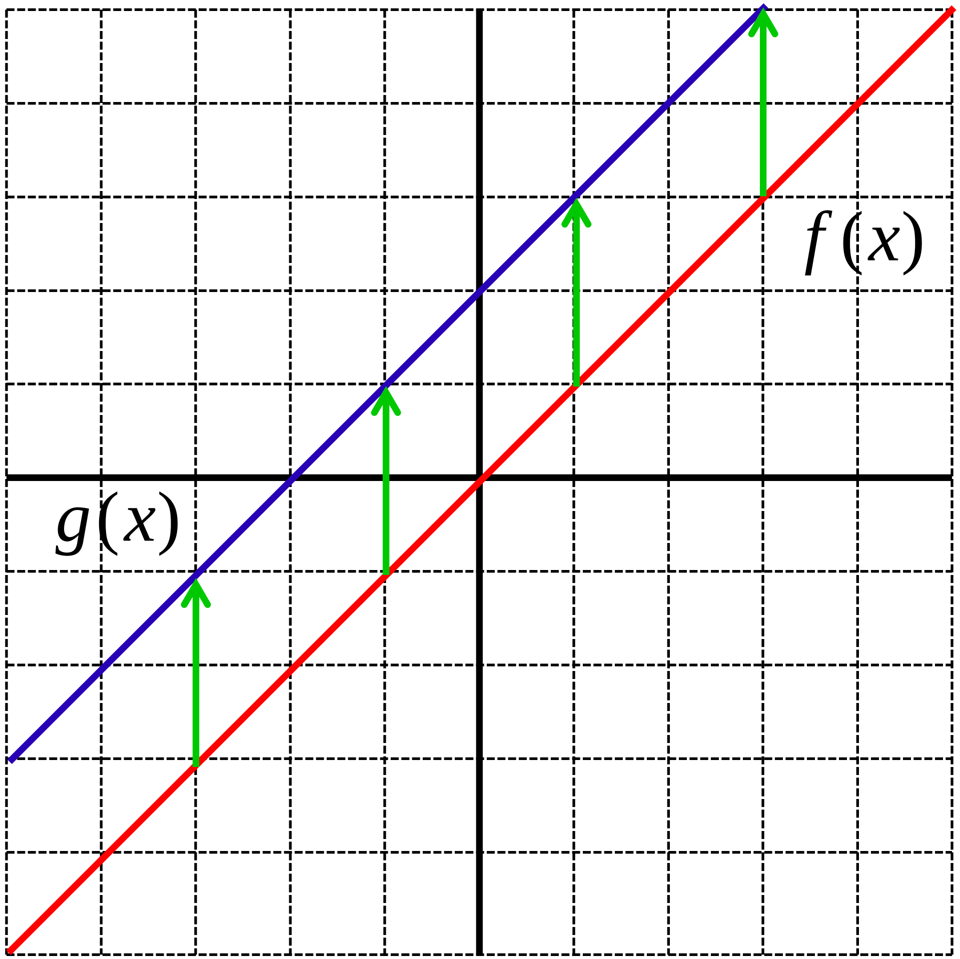
A. $g(3) = f(3)$

B. $g(3) = f(3) + 2$

C. $g(3) = f(3) - 2$

D. $g(3) = 2f(3)$

E. $g(3) = \frac{1}{2} f(3)$



Solution

Answer: B

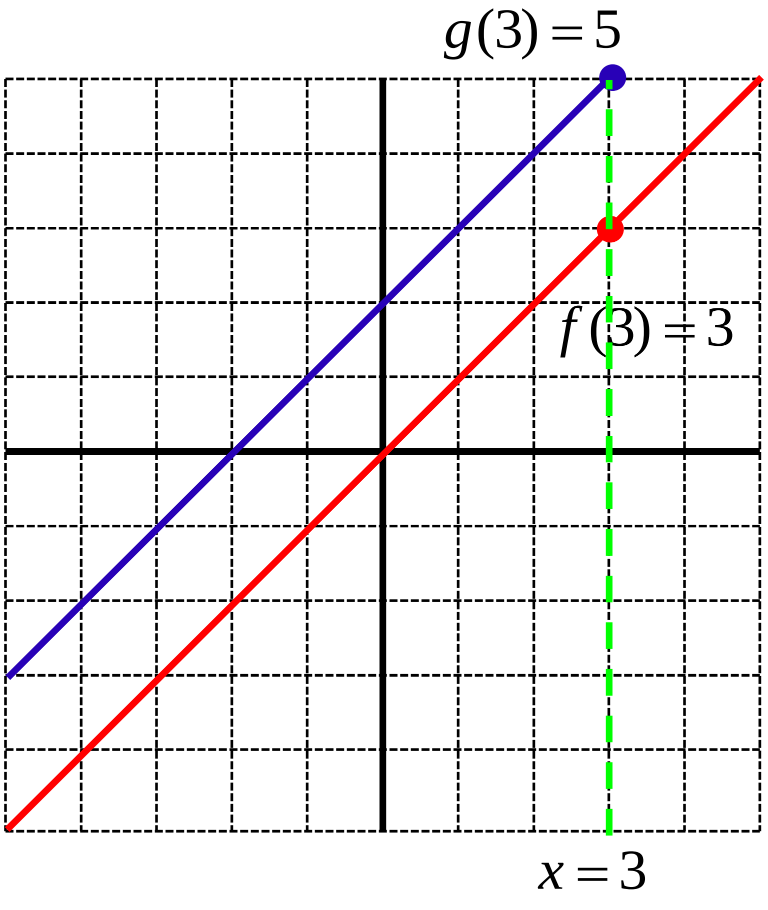
Justification: Use the graph to find the values of $f(3)$ and $g(3)$:

$f(3) = 3 \qquad g(3) = 5$

$g(3)$ can be expressed in terms of $f(3)$:

$g(3) = 5 = 3 + 2 = f(3) + 2$

Notice on the graph that $g(3)$ is 2 units above $f(3)$.



Transformation of Lines II

The graph of $g(x)$ is formed by taking every value of $f(x)$ and translating (moving) it vertically upwards by 2 units.

What is $g(x)$ written in terms of $f(x)$?

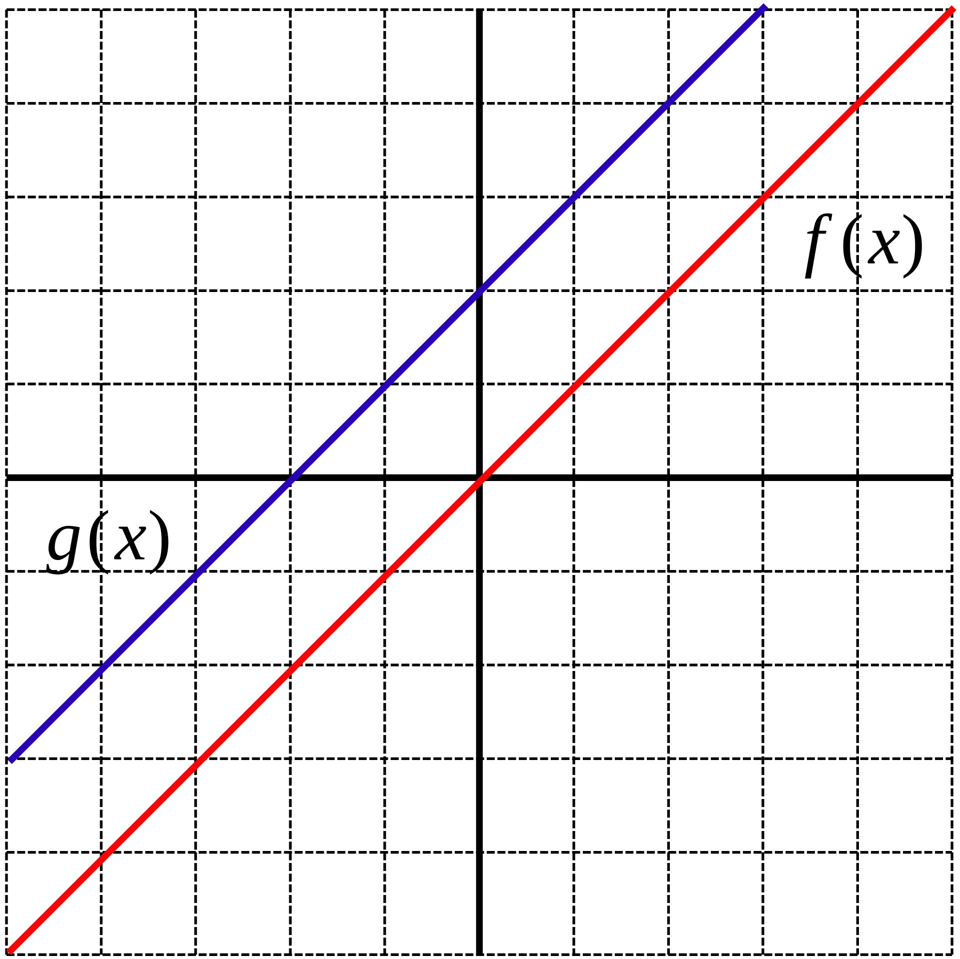
A. $g(x) = 2f(x)$

B. $g(x) = f(x) + 2$

C. $g(x) = f(x) - 2$

D. $g(x) = f(x+2)$

E. $g(x) = f(x-2)$



Solution

Answer: B

Justification: The equation of the linear functions are:

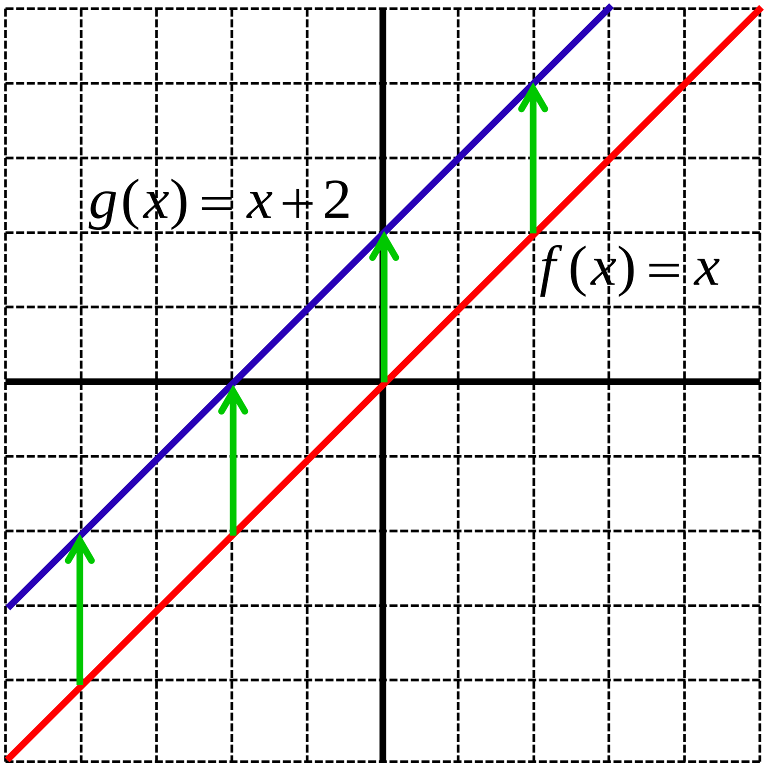
$$f(x) = x$$

$$g(x) = x + 2$$

Substituting $f(x) = x$ into $g(x)$ will express $g(x)$ in terms of $f(x)$:

$$g(x) = x + 2 = f(x) + 2$$

The function $g(x)$ always returns a result that is 2 larger than $f(x)$.



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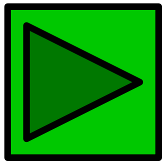
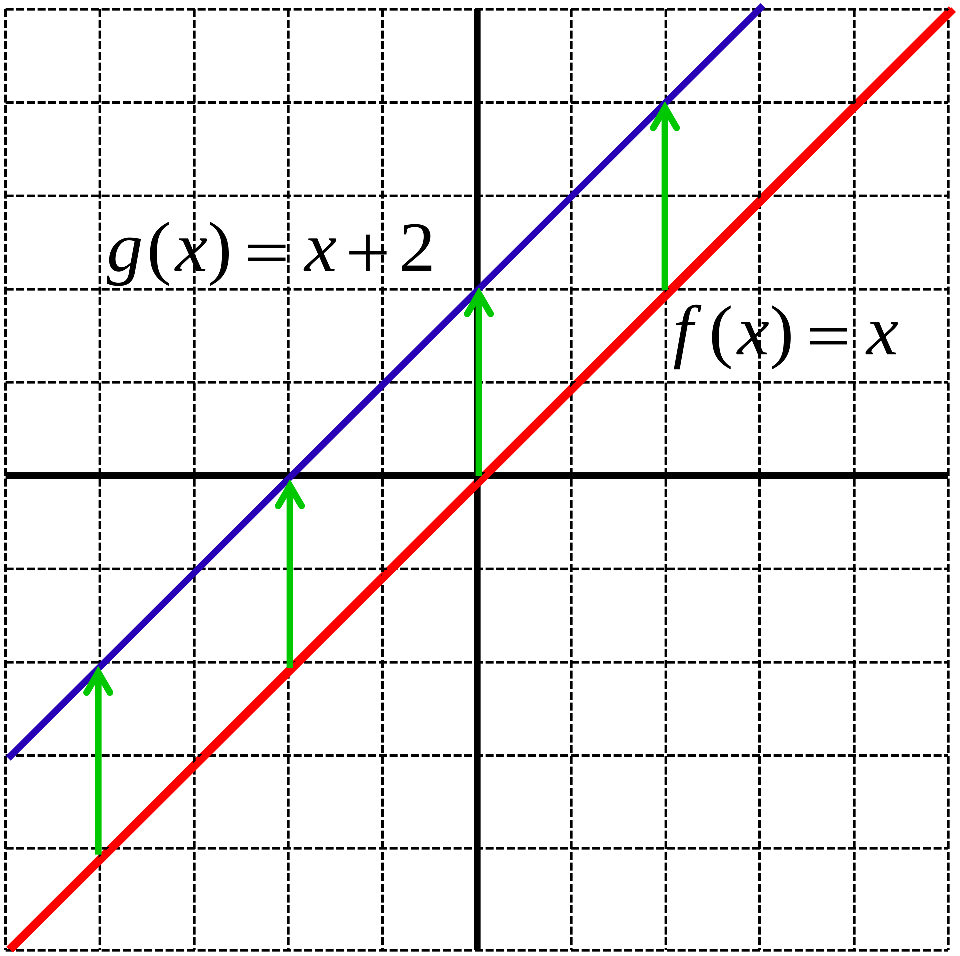




Translate up 2 units

$$g(x) = f(x) + 2$$

Transformation of Lines III

If $f(a) = g(-3)$, what is the value of a?

Press for hint

Find the value of g(-3). Where does $f(x)$ also equal this value?

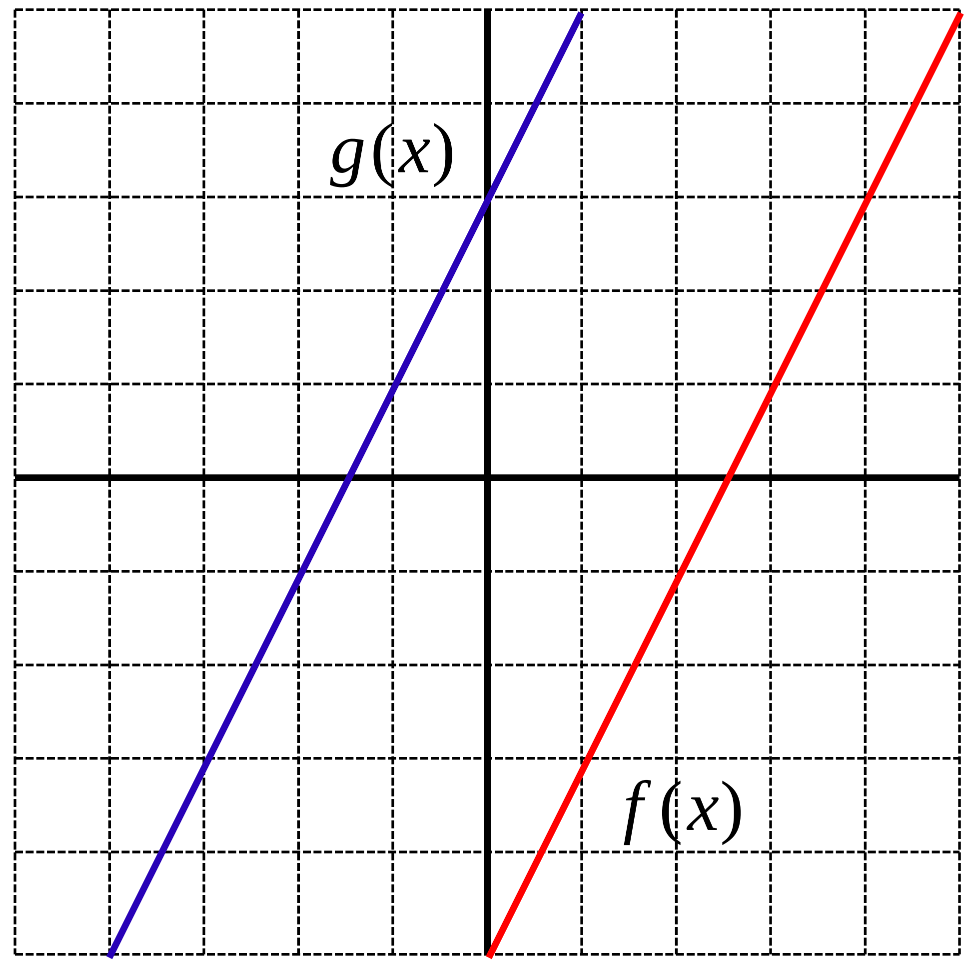
A. $a = 3$

B. $a = 1$

C. $a = -1$

D. $a = -3$

E. None of the above



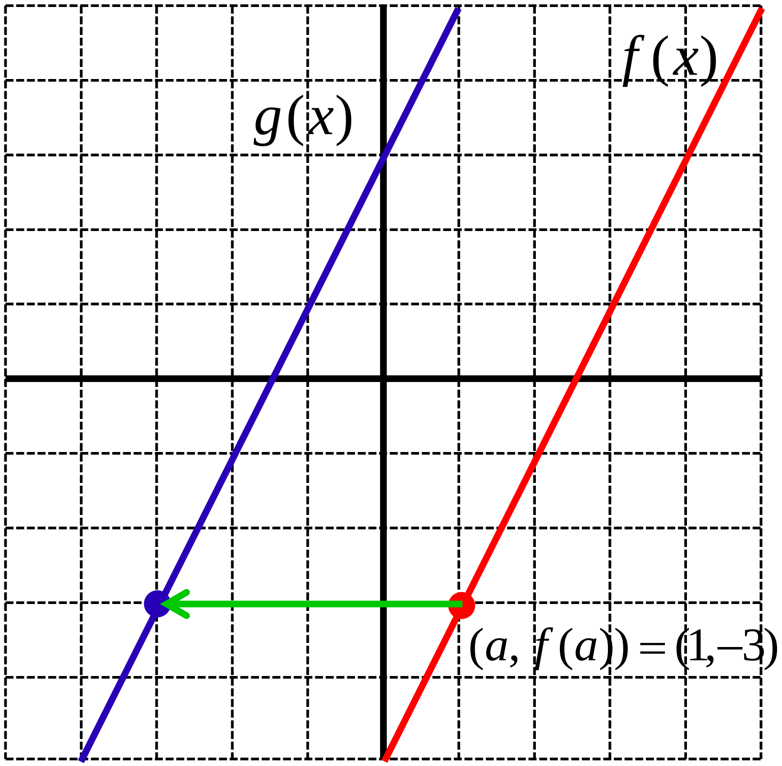
Solution

Answer: B

Justification: The value of $g(-3)$ is -3. The function $f(x)$ only maps to -3 when $f(1)$.

$$g(-3) = f(1) = -3,$$

Notice that since $g(-3)$ and $f(1)$ have the same value, both points lie on the same horizontal line.



Transformation of Lines IV

The function $g(x)$ represents the function $f(x)$ after it has been translated horizontally by -4 units (to the left).

What is $g(x)$ in terms of $f(x)$?

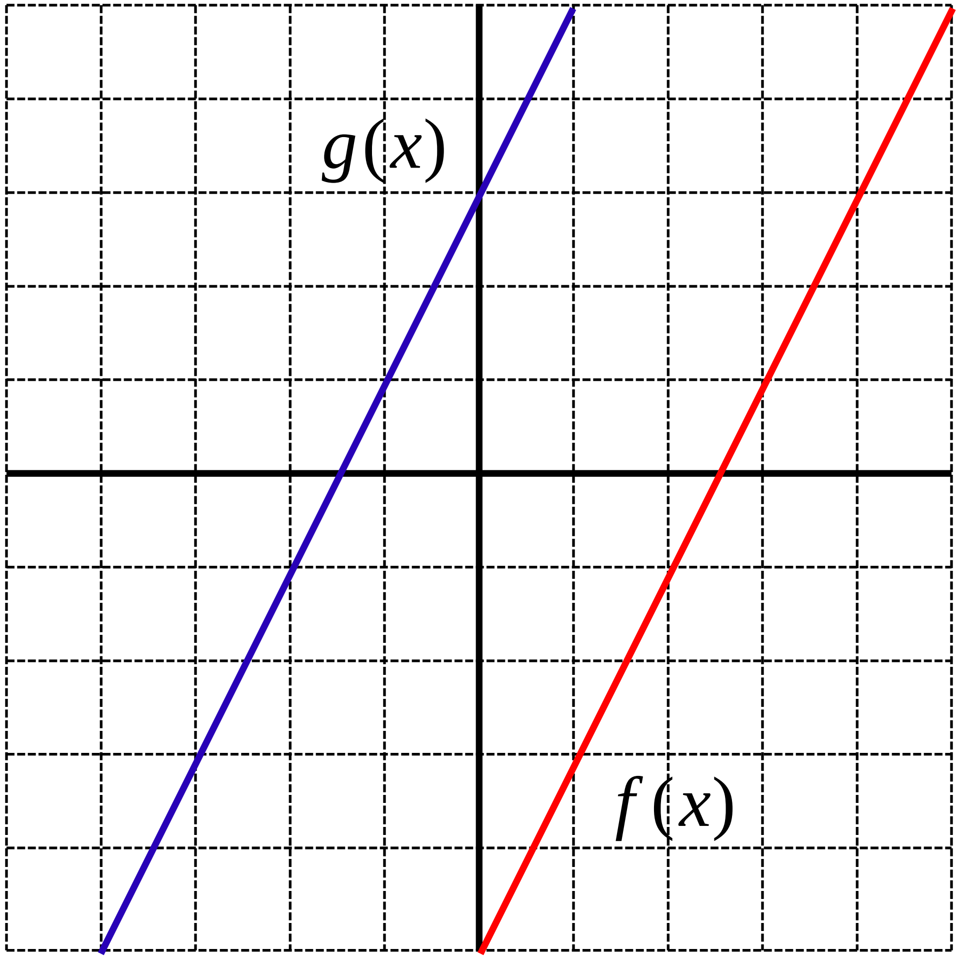
A. $g(x) = f(x) + 4$

B. $g(x) = f(x) - 4$

C. $g(x) = f(x+4)$

D. $g(x) = f(x-4)$

E. None of the above



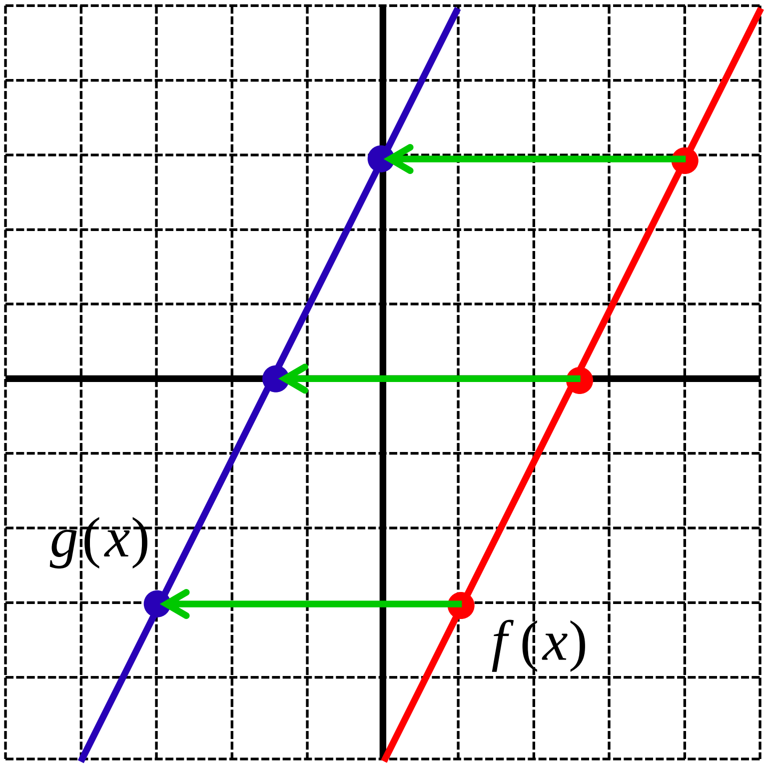
Solution

Answer: C

Justification: Consider points on each function that lie on the same horizontal line. The x-value of f at these points is always 4 greater than those of g.

Recall that $g(-3) = f(1) = -3$.

The function $g(x)$ is therefore equivalent to $f(x+4)$. This shifts the graph of $f(x)$ to the left by four units.



Explanation continues on the next slide

Solution Cont' d

The equations of the two lines are:

$$f(x) = 2x - 5$$

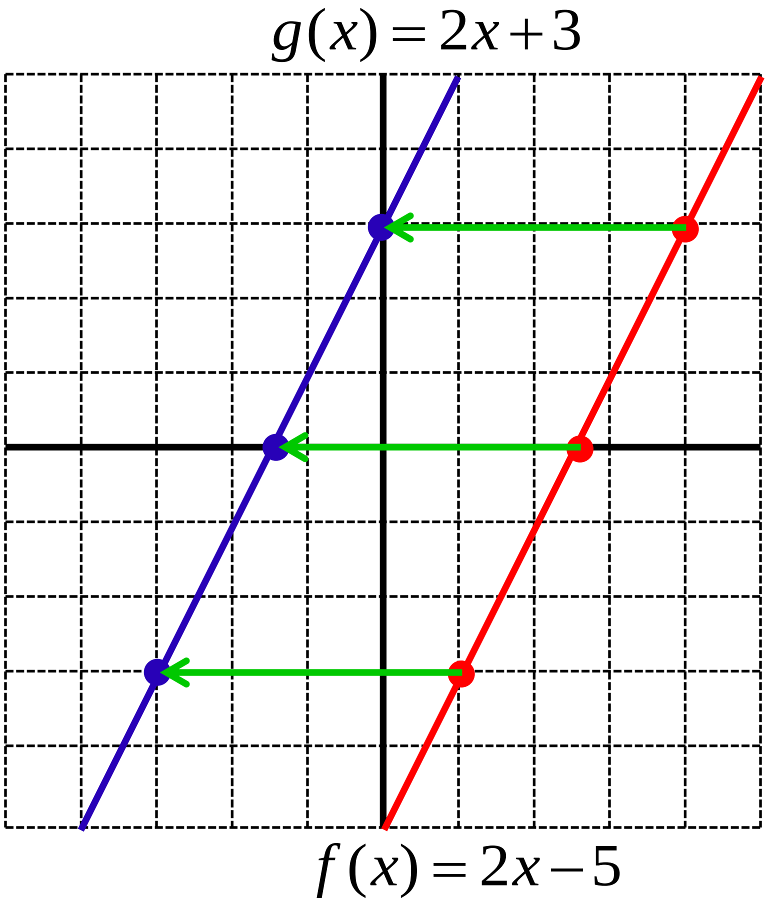
$$g(x) = 2x + 3$$

We can check that $g(x) = f(x+4)$ by substituting x + 4 into $f(x)$:

$$f(x+4) = 2(x+4) - 5$$

$$= 2x + 3$$

$$= g(x)$$



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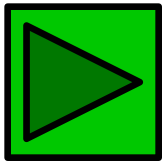
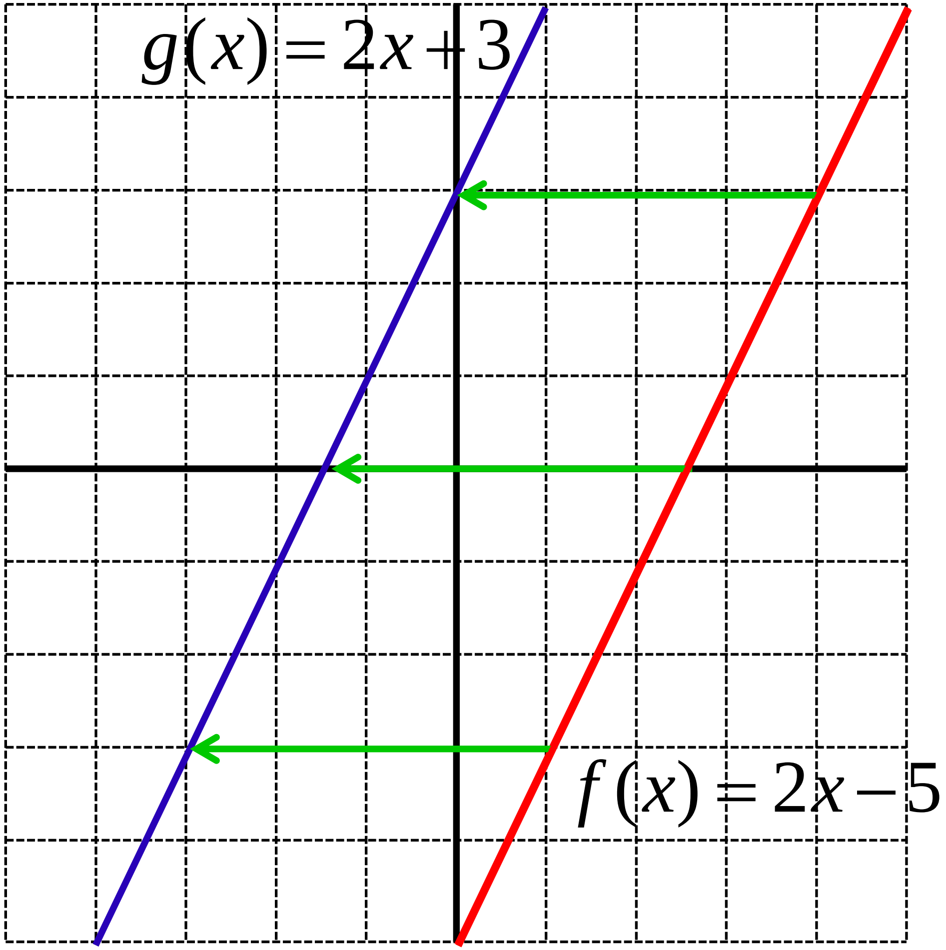


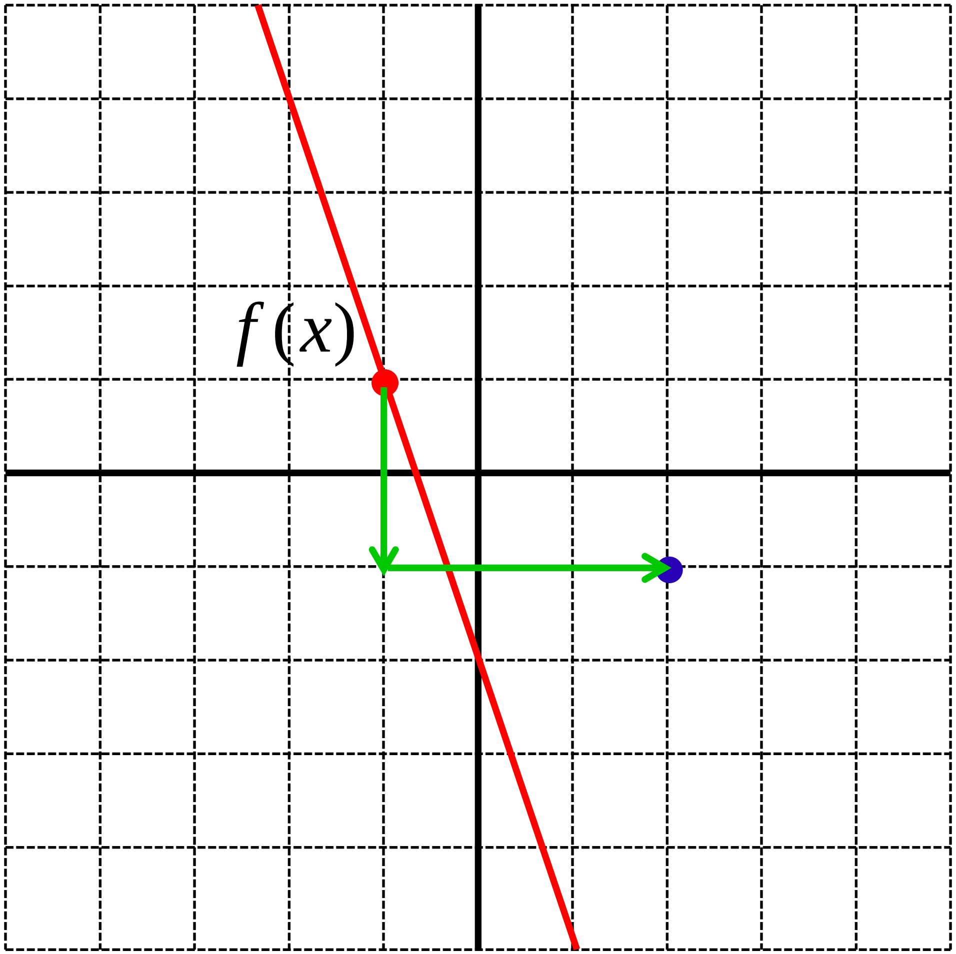
Translate left 4 units

$$g(x) = f(x+4)$$



Transformation of Lines V

The function $f(x)$ is translated 2 units down and 3 units right. What is the equation of this new function, $g(x)$, in terms of $f(x)$?



A. $g(x) = f(x+3)+2$

B. $g(x) = f(x-3)+2$

C. $g(x) = f(x+3)-2$

D. $g(x) = f(x-3)-2$

E. None of the above

Solution

Answer: D

Justification: Start with $y = f(x)$.

Translate y 2 units down by replacing $f(x)$ with $f(x)-2$:

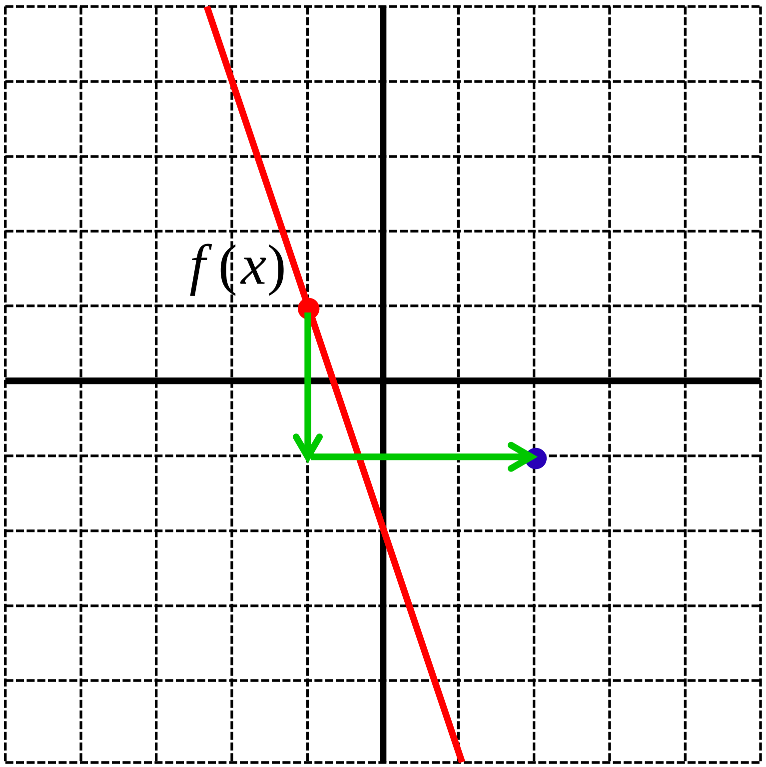
$$y = f(x)$$

$$y = f(x) - 2 \quad \textit{2 units down}$$

Translate y 3 units to the right by replacing $f(x)$ with $f(x-3)$:

$$y = f(x) - 2$$

$$g(x) = f(x-3) - 2 \quad \textit{3 units right}$$



Review questions 2 and 4 if you got the signs incorrect.

Transformation of Lines VI

The function $f(x)$ is translated 2 units down and 3 units right. What is the equation of this new function, $g(x)$, in terms of x?

$$g(x) = f(x-3)-2$$

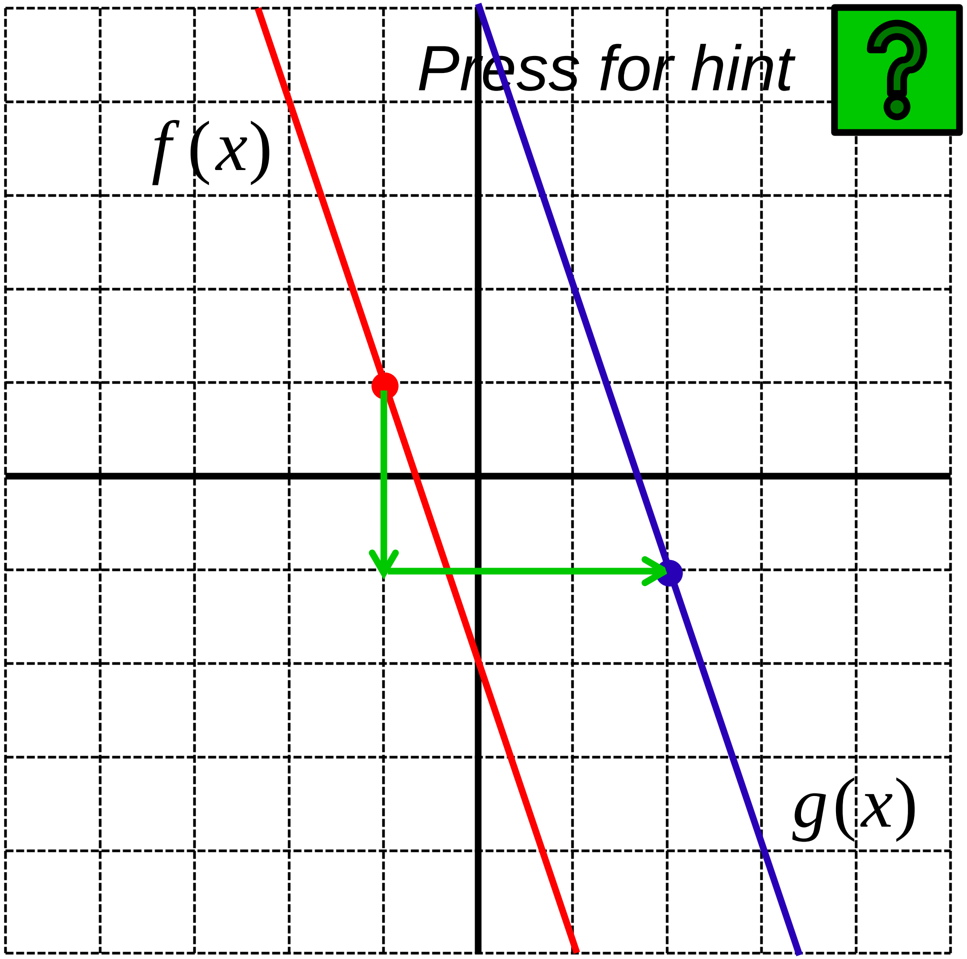
A. $g(x) = -3x+7$

B. $g(x) = -3x+5$

C. $g(x) = -3x-2$

D. $g(x) = -3x-4$

E. $g(x) = -3x-5$



Solution

Answer: D

Justification: Graph the translated line and find its equation. One way to do this is translating two points on f to find two points on g. For example:

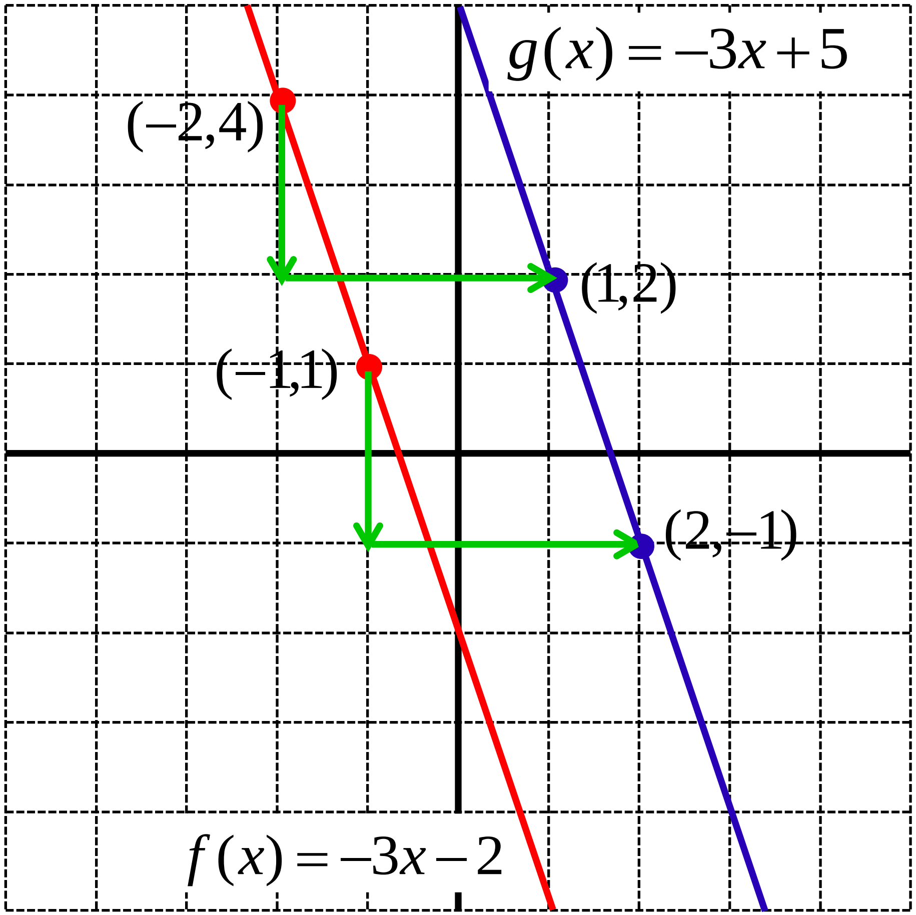
$$(-1,1) \to (2,-1) \qquad (-2,4) \to (1,2)$$

The slope and y-intercept are:

$$m = \frac{2-(-1)}{1-2} = -3 \qquad b = 5$$

The equation of the line is:

$$g(x) = -3x + 5$$



The solution is continued on the next slide.

Solution Cont'd

For students that are comfortable working with composite functions, we can double check our answer.

First find the equation of the line for $f(x)$:

$$f(x) = -3x - 2$$

Since we know from the previous question that:

$$g(x) = f(x-3) - 2$$

We can now solve for $g(x)$:

$$g(x) = f(x-3) - 2$$

$$g(x) = [-3(x-3) - 2] - 2 \quad \text{since} \quad f(x-3) = -3(x-3) - 2$$

$$g(x) = -3x + 9 - 2 - 2$$

$$g(x) = -3x + 5$$

Transformation of Lines VII

The function $f(x) = mx$ is shifted by p units horizontally and q units vertically. What is the y-intercept of the new line?

A. $mp + q$

B. $-mp + q$

C. $mp - q$

D. $-mp - q$

E. q

Press for hint

Horizontal translation by p units:

$$g(x) = f(x - p)$$

Vertical translation by q units:

$$g(x) = f(x) + q$$

Solution

Answer: B

Justification: After the line has been translated, we get the equation:

$$g(x) = m(x - p) + q$$

Positive values of p shift the graph right, while negative values shift the graph left.

Positive values of q shift the graph up, while negative values shift the graph down.

We can get the y-intercept by expanding the equation:

$$g(x) = mx - mp + q$$

This is in the form $y = mx + b$, where the y-intercept is $b = -mp + q$.

Transformation of Lines VIII

The function $f(x)$ is reflected over the y-axis as shown to produce $g(x)$. What is the relationship between $f(x)$ and $g(x)$?

Press for hint

Compare $f(2)$ and $g(-2)$.

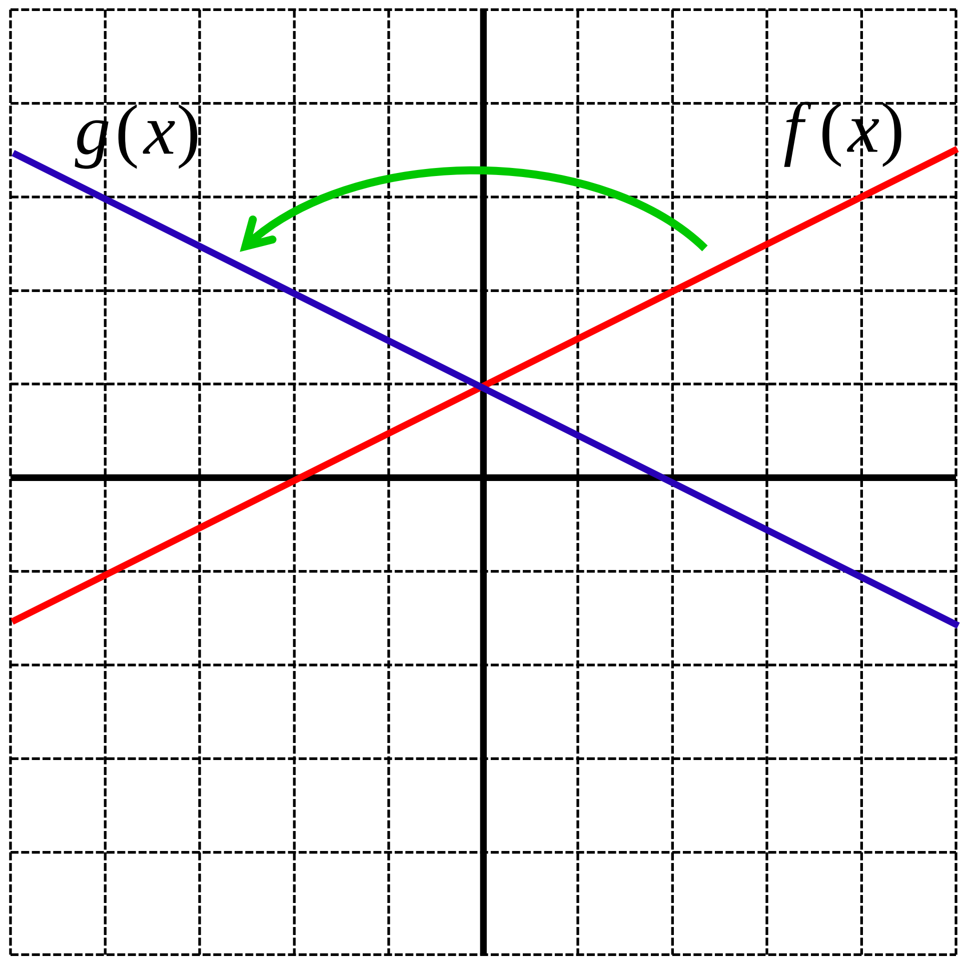
A. $g(x) = -f(x)$

B. $g(x) = f(-x)$

C. $g(x) = -f(-x)$

D. $g(x) = f(x) - x$

E. None of the above



Solution

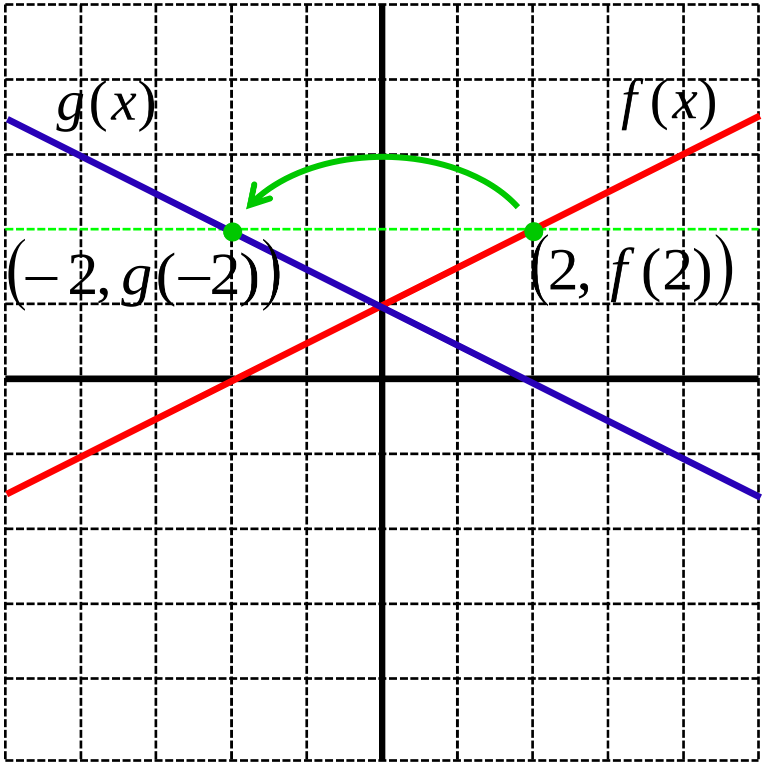
Answer: B

Justification: Consider what happens to $f(2)$ when it is reflected across the y-axis. The value of $f(2)$ is equivalent to the value of $g(-2)$.

To produce $g(x)$, every point in $f(x)$ has its x-coordinate multiplied by -1. This is represented by:

$$g(x) = f(-x)$$

Notice that $\left(2, f(2)\right)$ and $\left(-2, g(-2)\right)$ are equal distances from the y-axis, and on the same horizontal line.



Animation

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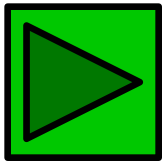
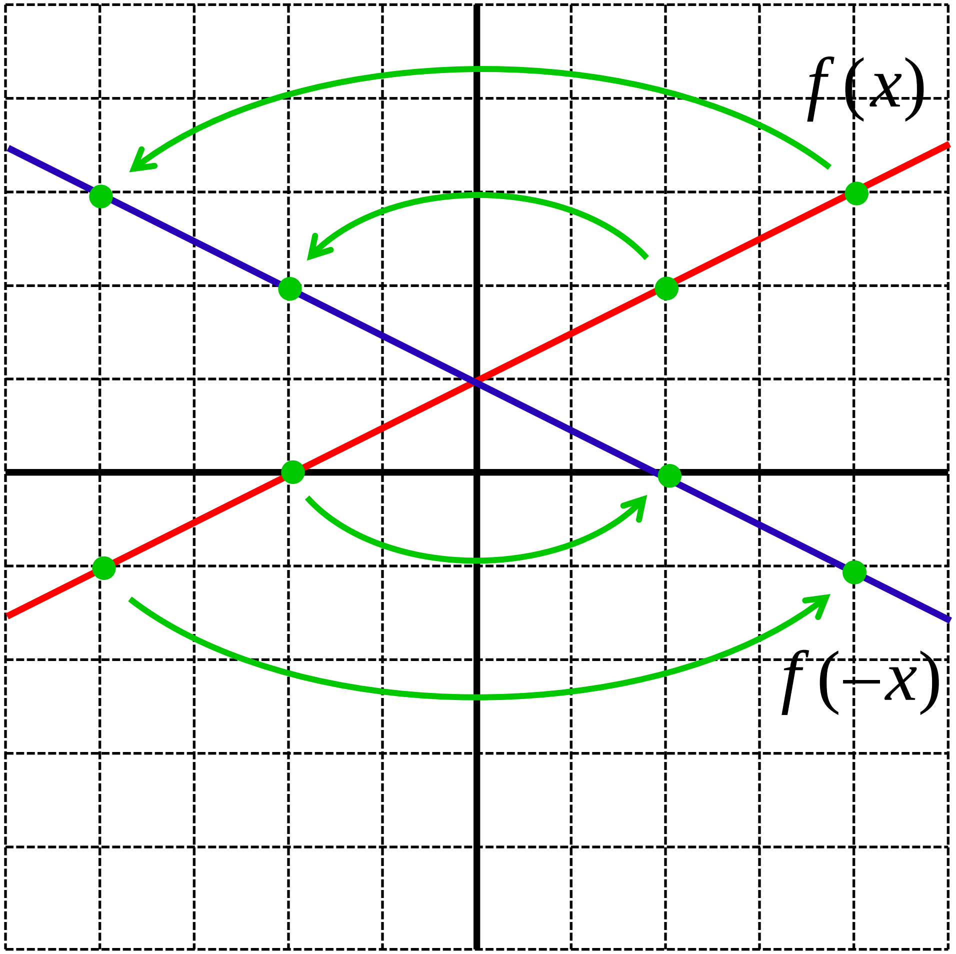




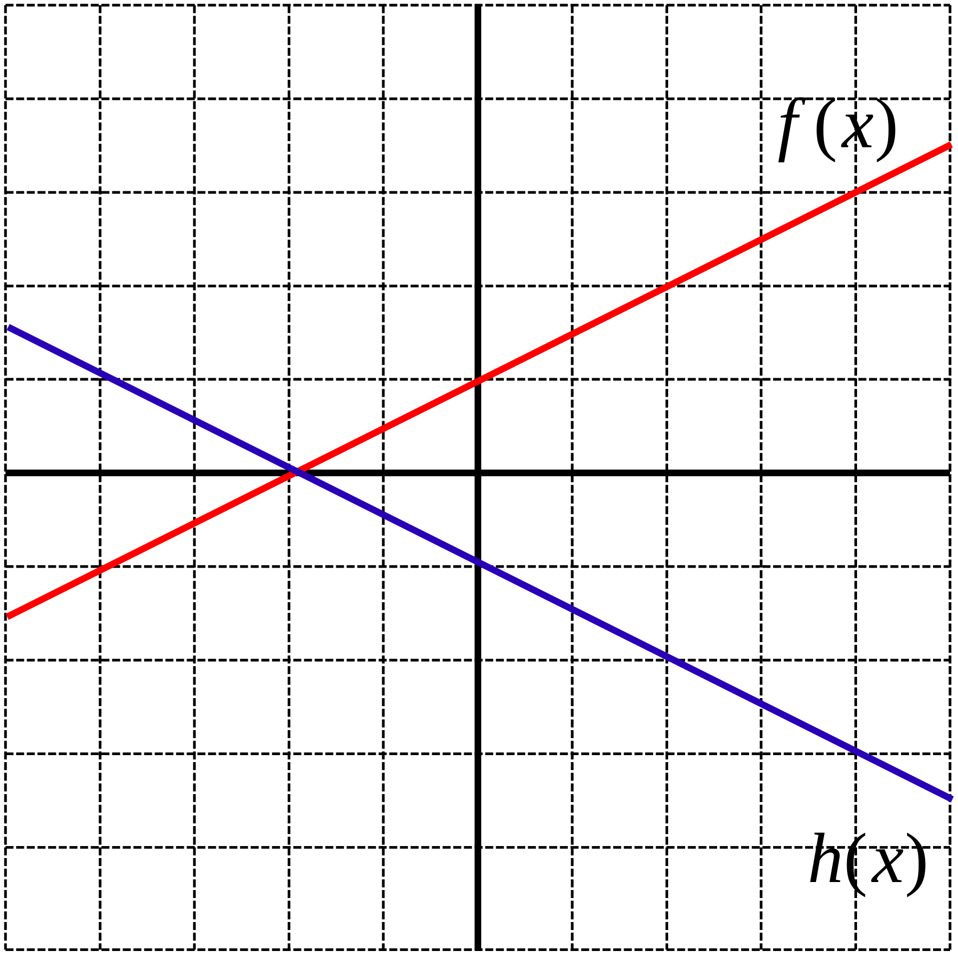
Reflection across y-axis

$$g(x) = f(-x)$$

Transformation of Lines IX

The function $f(x)$ is now reflected across the x-axis to form $h(x)$. How do the slope and y-intercept of $h(x)$ compare to those of $f(x)$?

	Slope	Y-intercept
A.	Change sign	Change sign
B.	Change sign	No change
C.	No change	Change sign
D.	No change	No change



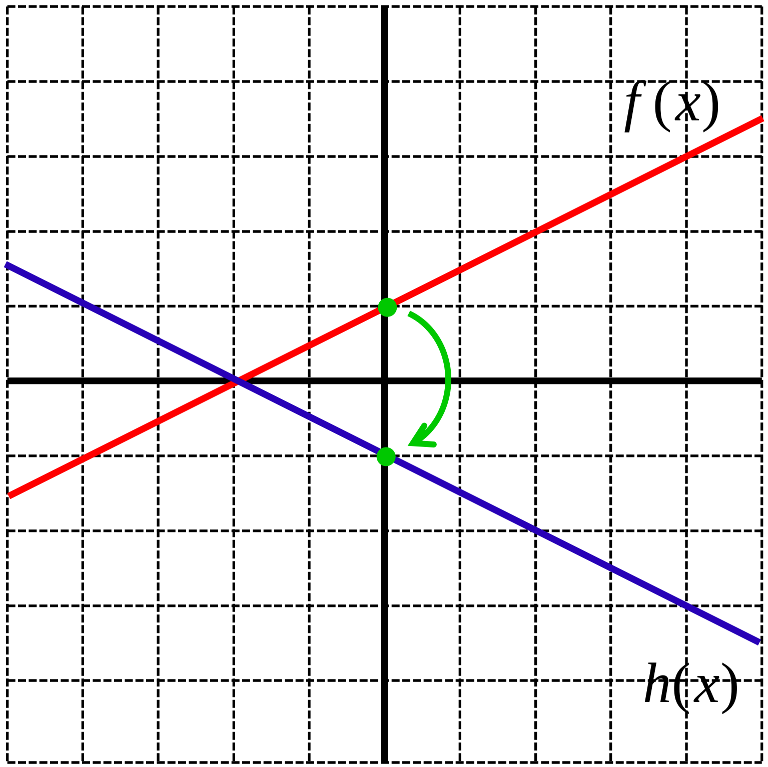
Solution

Answer: A

Justification: When reflecting points of $f(x)$ across the x-axis, the y-coordinate is multiplied by -1.

If the y-intercept of $f(x)$ is (0, 1), the y-intercept of $h(x)$ is (0, -1). The y-intercept of $f(x)$ changes sign.

From the graph you can also see that the slope of the line changes sign, from 0.5 to -0.5. How can you prove this to yourself?



Reflection across x-axis:
$h(x) = -f(x)$

Transformation of Lines X

The function $f(x)$ is reflected across the x-axis to form $g(x)$. What is the equation of $g(x)$?

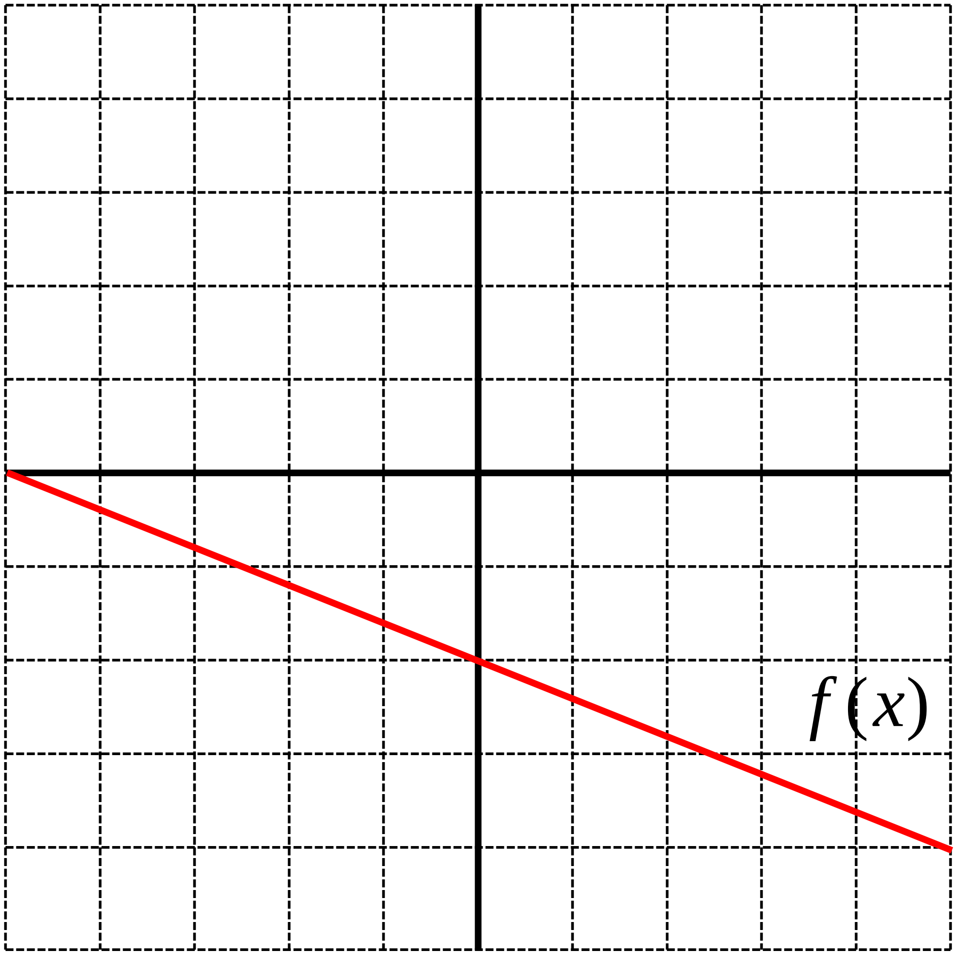
A. $g(x) = \frac{5}{2} - 2$

B. $g(x) = \frac{5}{2} + 2$

C. $g(x) = \frac{2}{5} - 2$

D. $g(x) = \frac{2}{5} + 2$

E. $g(x) = -\frac{2}{5} - 2$

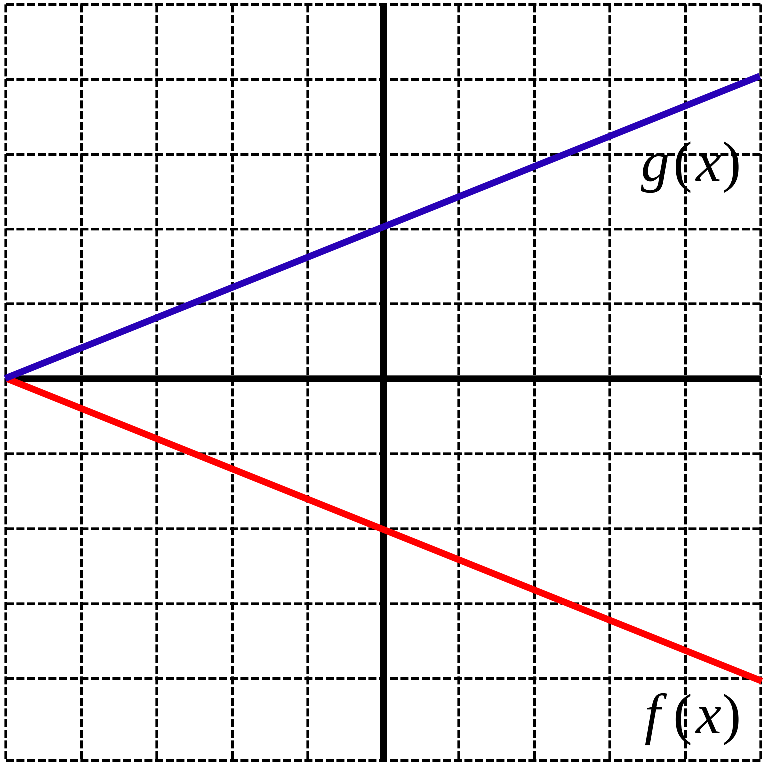


Solution

Answer: D

Justification: The equation of $f(x)$ is:

$$f(x) = -\frac{2}{5} - 2$$



After reflecting across the x-axis:

$$g(x) = -f(x)$$

Substituting in $f(x)$ into $g(x)$ gives:

$$g(x) = -\left(-\frac{2}{5} - 2\right)$$

$$g(x) = \frac{2}{5} + 2$$

Notice both the slope and y-intercept change sign

Transformation of Lines XI

Consider the two functions shown in the graph. What is the relationship between $f(x)$ and $g(x)$?

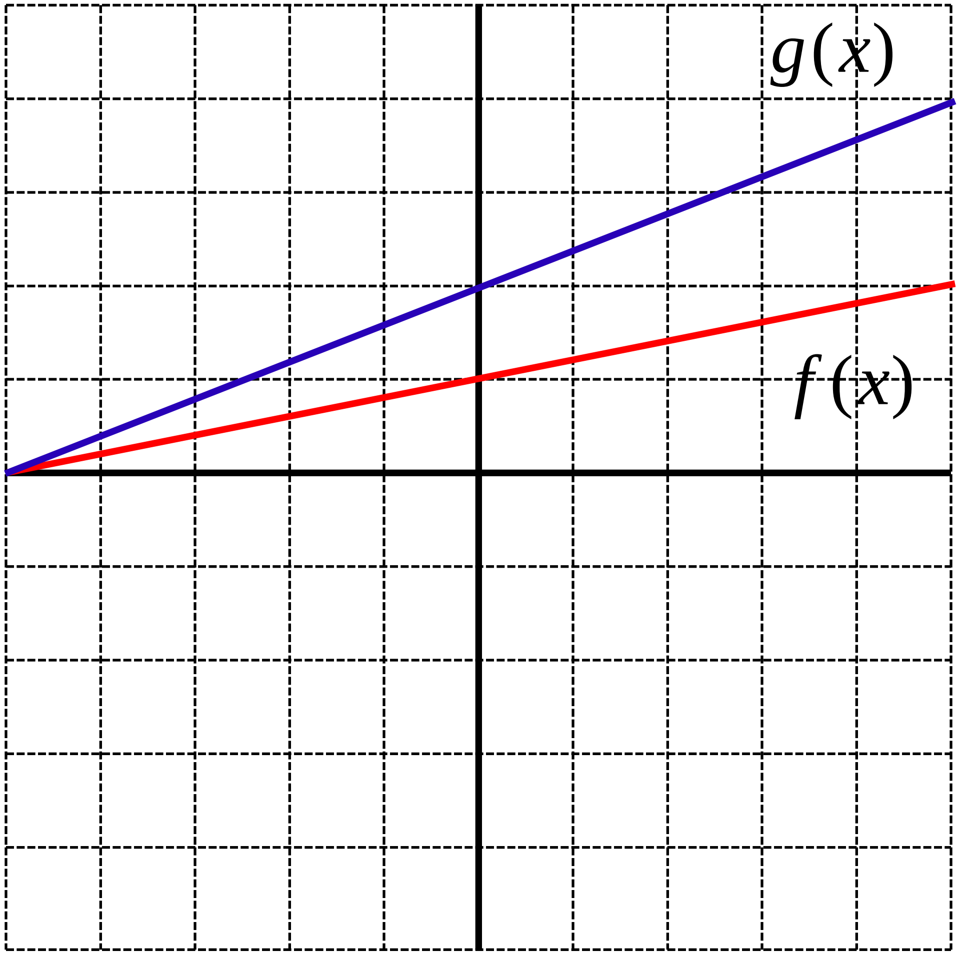
A. $g(x) = f(x) + x$

B. $g(x) = f(\frac{1}{2}x)$

C. $g(x) = f(2x)$

D. $g(x) = \frac{1}{2}f(x)$

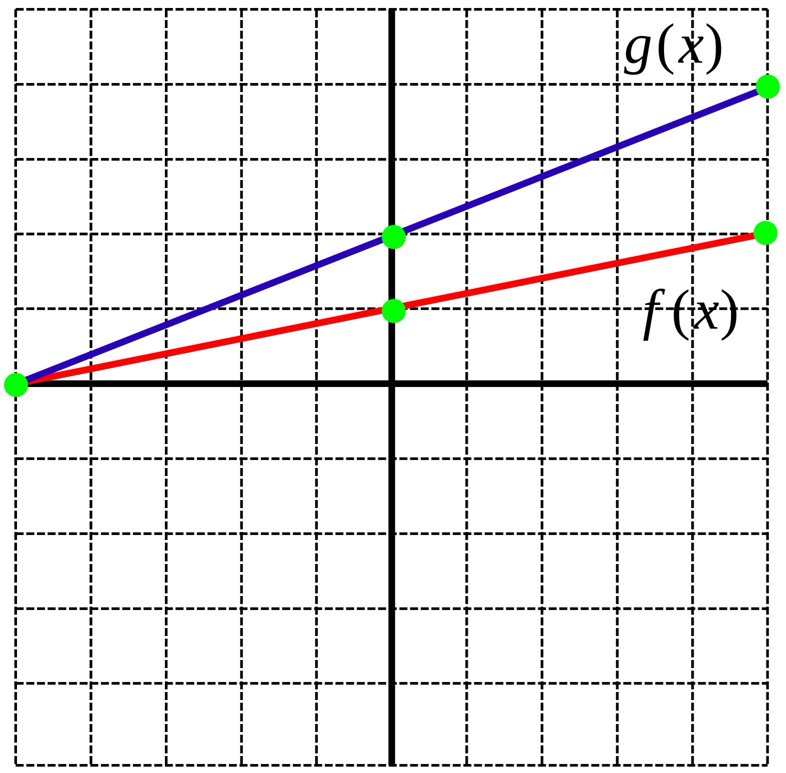
E. $g(x) = 2f(x)$



Solution

Answer: E

Justification: Observe that the difference between $g(x)$ and $f(x)$ changes for different values of x. The slope of $g(x)$ is larger than $f(x)$.



Compare the values of $g(x)$ and $f(x)$:

x	$f(x)$	$g(x)$
-5	0	0
0	1	2
5	2	4

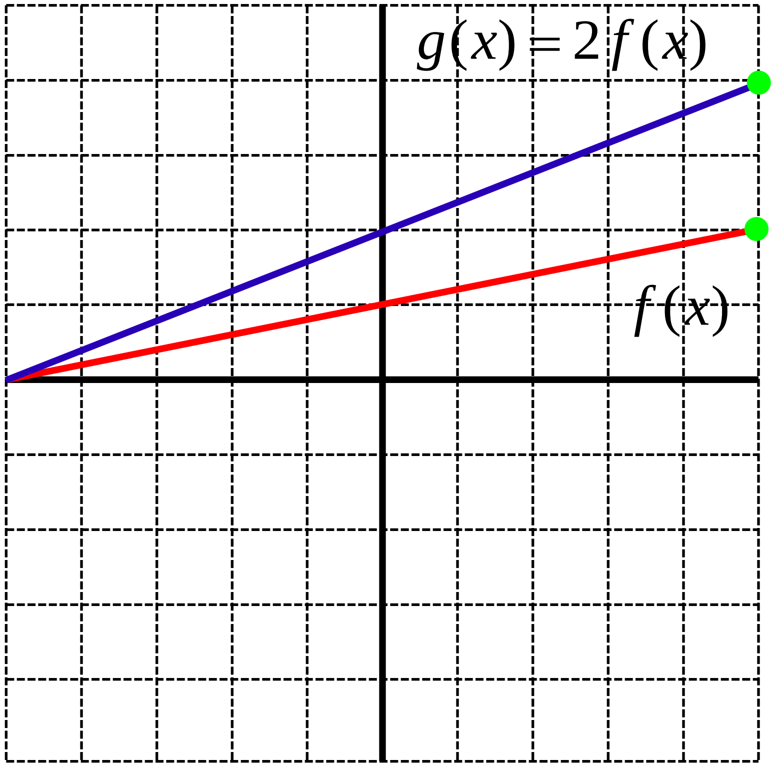
Solution continues on the next slide

Solution Cont'd

The value of $g(x)$ is always twice that of $f(x)$:

$$g(x) = 2f(x)$$

The function $2f(x)$ is a vertical expansion of $f(x)$ by a factor of 2. The graph of $f(x)$ is stretched vertically such that all values are doubled.



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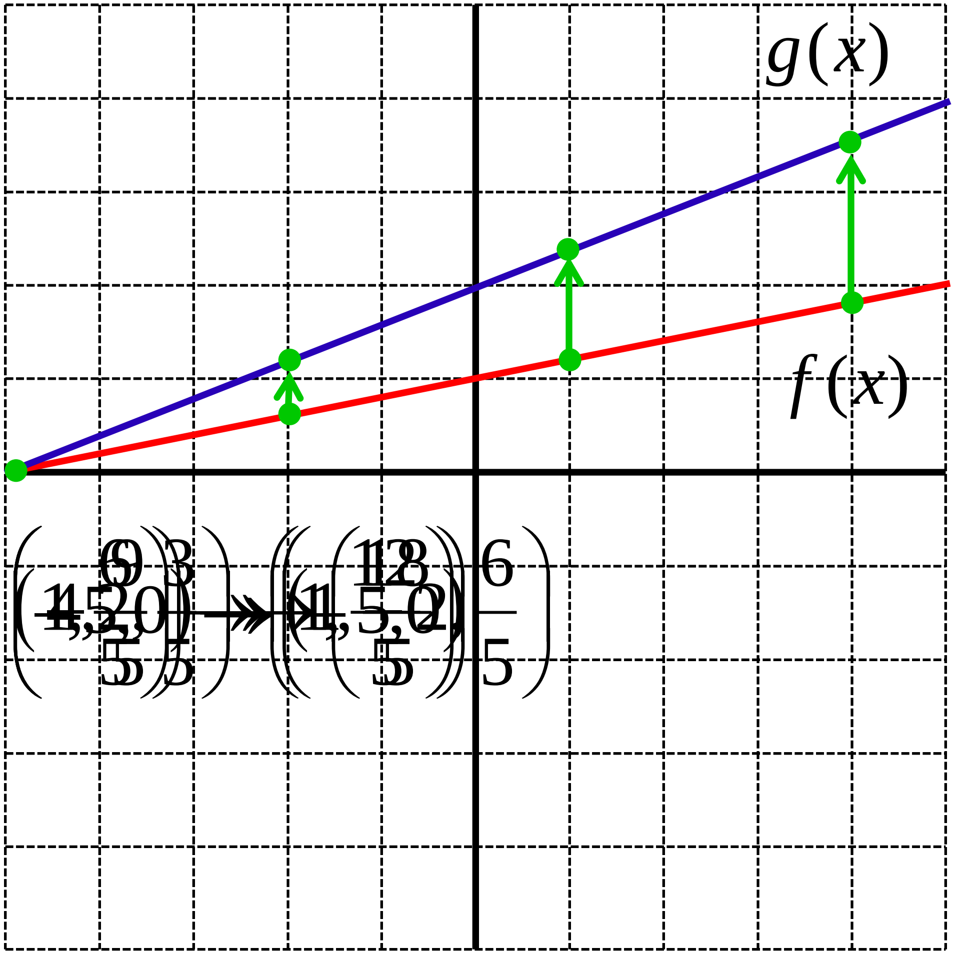


Vertical expansion by 2

$$g(x) = 2f(x)$$

Transformation of Lines XII

The graph of $f(x)$ is shown in red. The graph is compressed horizontally by a factor of $\frac{3}{4}$.

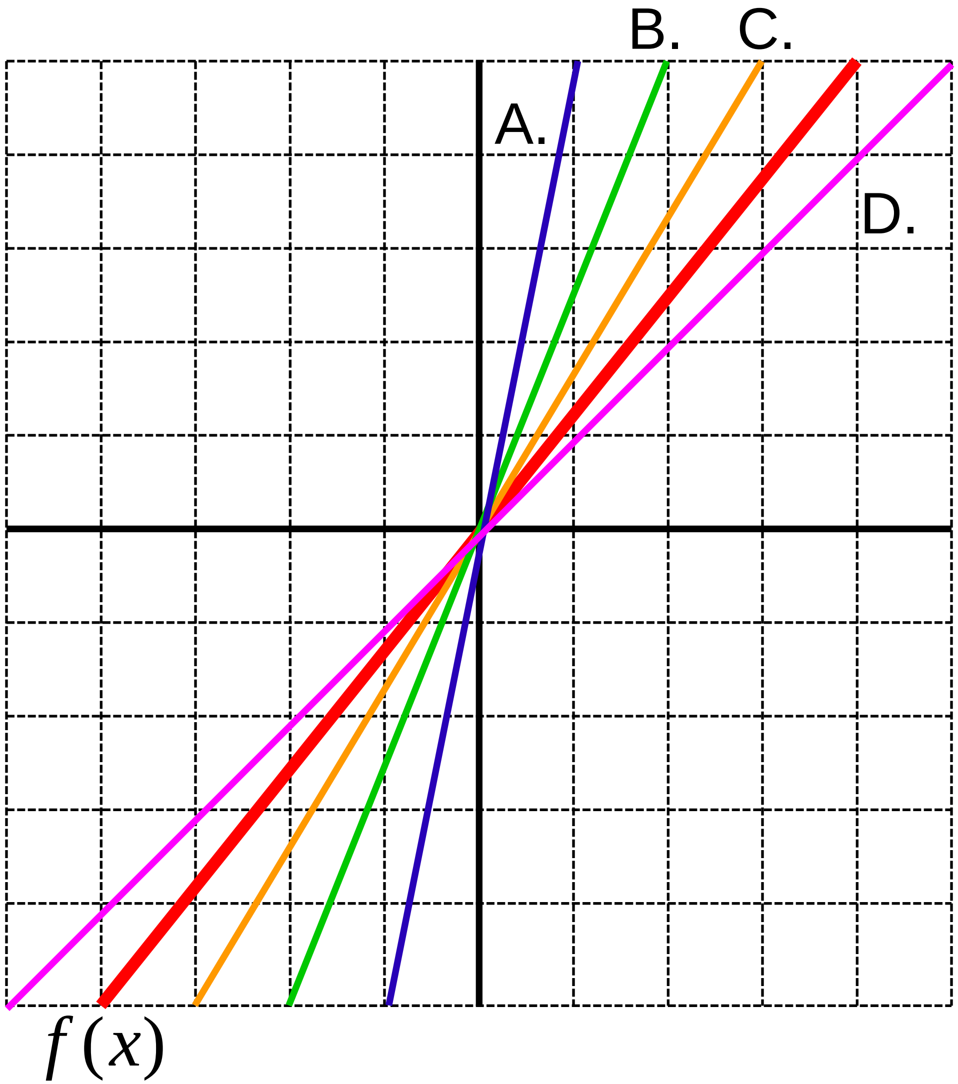
Which graph represents $f(x)$ after the transformation?

A. Blue graph

B. Green graph

C. Orange graph

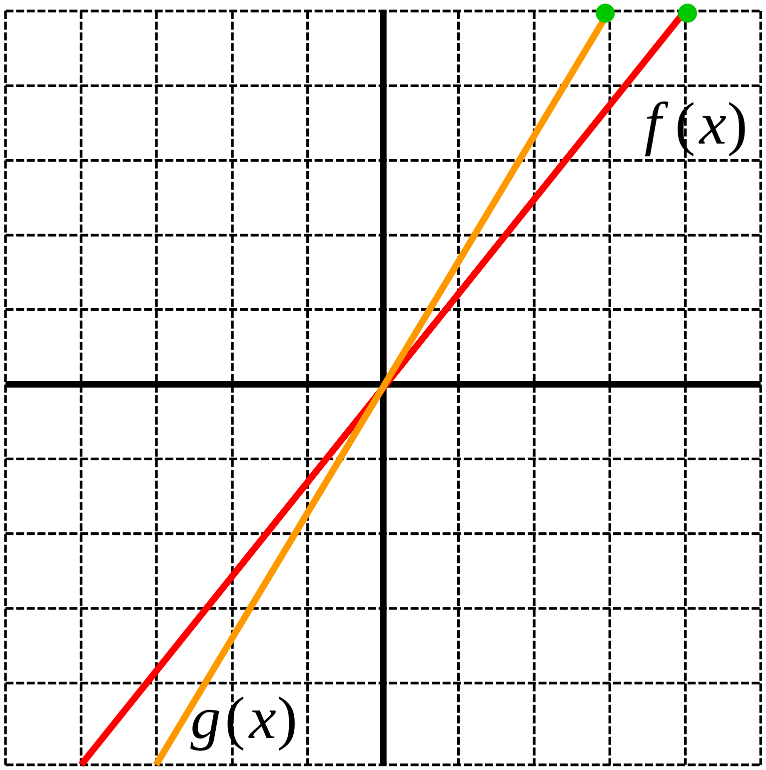
D. Purple graph



Solution

Answer: C

Justification: Consider the point $(4,5)$ which lies on the function f. If the function f is horizontally compressed by a factor of $\frac{3}{4}$, then the x-coordinate on g should be smaller by a factor of $\frac{3}{4}$. The y-coordinate of the point does not change, since we are compressing <u>horizontally</u>.



Therefore, the point $(3, 5)$ should be on g, which lies only on the line C (orange).

Transformation of Lines XIII

The function f is stretched by a factor of k, resulting in the new function g. If $k > 1$, the stretch is called an expansion. If $0 < k < 1$, the stretch is called a compression. What is the function g in terms of f?

A. $g(x) = kf(x)$

B. $g(x) = \frac{1}{k} f(x)$

C. $g(x) = f(kx)$

D. $g(x) = f\left(\frac{x}{k}\right)$

E. None of the above

Solution

Answer: D

Justification: A horizontal stretch by a factor of k means that if point $(a, f(a))$ is on f, then the point $(ka, f(a)) = (ka, g(ka))$ is on g.

Notice that when $k > 1$, g is horizontally expanded (the x-coordinate is farther away from the origin). When $0 < k < 1$, g is horizontally compressed (the x-coordinate is closer to the origin).

If we equate the two y-coordinates of the points $(a, f(a))$ and $(ka, g(ka))$, we get: $f(a) = g(ka)$

To express the function g in terms of f, substitute $a = \frac{x}{k}$:

$$f\left(\frac{x}{k}\right) = g\left(k \cdot \frac{x}{k}\right)$$

$$f\left(\frac{x}{k}\right) = g(x) \longrightarrow \boxed{g(x) = f\left(\frac{x}{k}\right)}$$

Transformation of Lines XIV

The line $y = mx + b$ is stretched horizontally by a factor of k, where $k > 1$. What is the new slope and y-intercept of the line?

	Slope	Y-intercept
A.	km	kb
B.	$\frac{m}{k}$	$\frac{b}{k}$
C.	km	b
D.	$\frac{m}{k}$	b
E.	$-km$	b

For horizontal stretches by a factor of k,

$$g(x) = f\left(\frac{x}{k}\right)$$

Press for hint

Solution

Answer: D

Justification: Recall that horizontal stretches by a factor of k have the general form:

$$g(x) = f\left(\frac{x}{k}\right)$$

When $x = 0$, $g(0) = f\left(0/k\right) = f(0)$. This means that the y-intercept does not change after horizontal expansions (or compressions) for any function.

If $f(x) = mx + b$, then $g(x) = f\left(\frac{x}{k}\right) = m\left(\frac{x}{k}\right) + b = \frac{m}{k}x + b$.

Notice that the slope gets multiplied by a factor of $1/k$. This means that horizontal expansions (k > 1) decrease the magnitude of the slope, while compressions (0 < k < 1) increase the magnitude of the slope.

Transformation of Lines XV

Which of the following sets of transformations, when applied to the function $f(x) = x$, forms the function $g(x) = -2x + 4$?

A. Reflection across y-axis, horizontal compression by $\frac{1}{2}$, translate 4 units left
B. Reflection across y-axis, horizontal compression by $\frac{1}{2}$, translate 4 units right
C. Reflection across x-axis, vertical expansion by 2, translate 2 units left
D. Reflection across x-axis, vertical expansion by 2, translate 2 units right
E. Reflection across x-axis, horizontal compression by $\frac{1}{2}$, translate 4 units left

Solution

Answer: D

Justification: We begin with the function $f(x) = x$.

Since $-f(x) = f(-x) = -x$, a reflection in the x-axis or y-axis are the same. Note that this is only true for odd functions, when $-f(x) = f(-x)$

Vertical expansions by 2, $g(x) = 2f(x)$, and horizontal compression by $\frac{1}{2}$, $g(x) = f(2x)$, are also the same. Note that this is not generally true. It just happens to be the case for $f(x) = x$, as shown:

Vertical expansion by 2:

$$f(x) = -x$$
$$2f(x) = -2x$$

Horizontal compression by $\frac{1}{2}$:

$$f(x) = -x$$
$$f(2x) = -(2x)$$

Answer continues on the next slide

Solution Cont' d

Answer: D

Justification: Since reflections and horizontal/vertical stretches produce the same result in this case, only the different horizontal translations will determine the answer.

Starting with $f(x) = -2x$, we want to replace x with $x-k$ (a horizontal translation by k units) so that: $g(x) = f(x-k) = -2x+4$

Solve for *k*: $-2(x-k) = -2x+4$

$$k = 2$$

This corresponds to a translation to the right by 2 units, since *k* is positive. Only answer D has a translation to the right by 2 units, after the reflection and expansion by 2.

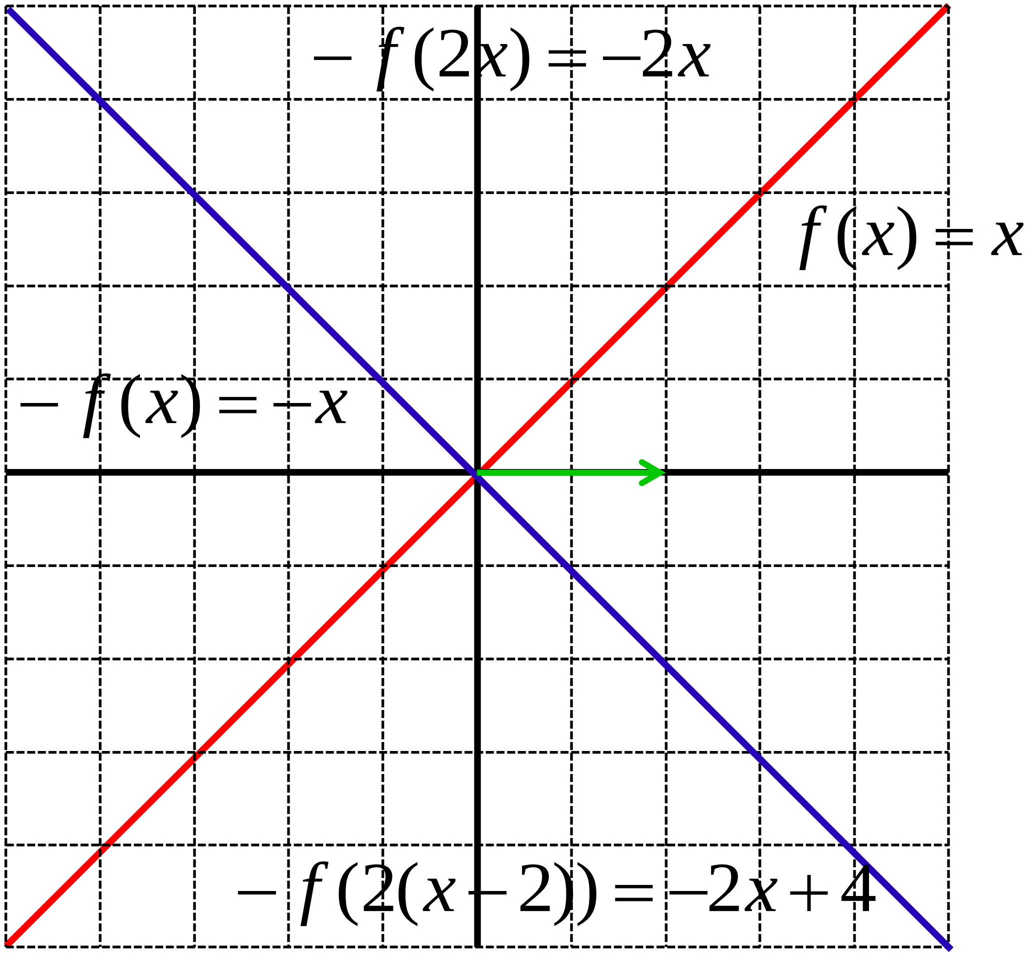
Animation for Question XV

Press to play animation:

Reflection across x-axis

Vertical expansion by 2

Translate 2 units right



Summary of Transformations

Vertical Translation $g(x) = f(x) + k$ k > 0, translate up k < 0 translate down	Horizontal Translation $g(x) = f(x - k)$ k > 0, translate right k < 0 translate left
Reflection across x-axis $g(x) = -f(x)$ y-values change sign	Reflection across y-axis $g(x) = f(-x)$ x-values change sign
Vertical stretches $g(x) = k \cdot f(x)$ k > 1, expansion 0 < k < 1 compression	Horizontal stretches $g(x) = f\left(\frac{x}{k}\right)$ k > 1, expansion 0 < k < 1 compression