a place of mind

FACULTY OF EDUCATION

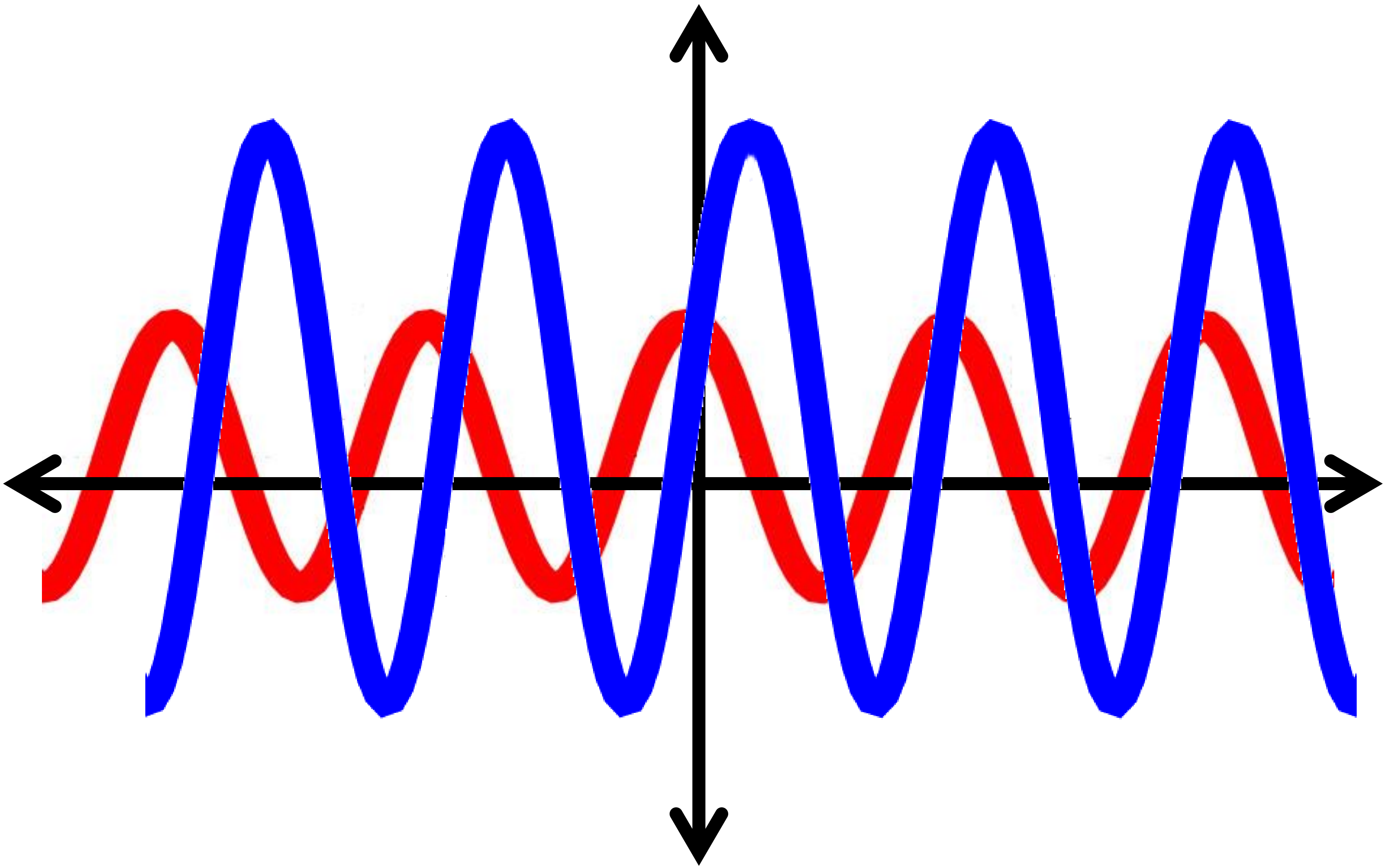
Department of
Curriculum and Pedagogy

Mathematics
Transformation on Trigonometric Functions

Science and Mathematics Education Research Group

Supported by UBC Teaching and Learning Enhancement Fund 2012-2014

Transformations on Trigonometric Functions



Summary of Transformations

Vertical Translation

$$g(x) = f(x) + k$$

k > 0, translate up

k < 0 translate down

Horizontal Translation

$$g(x) = f(x - k)$$

k > 0, translate right

k < 0 translate left

Reflection across x-axis

$$g(x) = -f(x)$$

y-values change sign

Reflection across y-axis

$$g(x) = f(-x)$$

x-values change sign

Vertical stretches

$$g(x) = k \cdot f(x)$$

k > 1, expansion

0 < k < 1 compression

Horizontal stretches

$$g(x) = f\left(\frac{x}{k}\right)$$

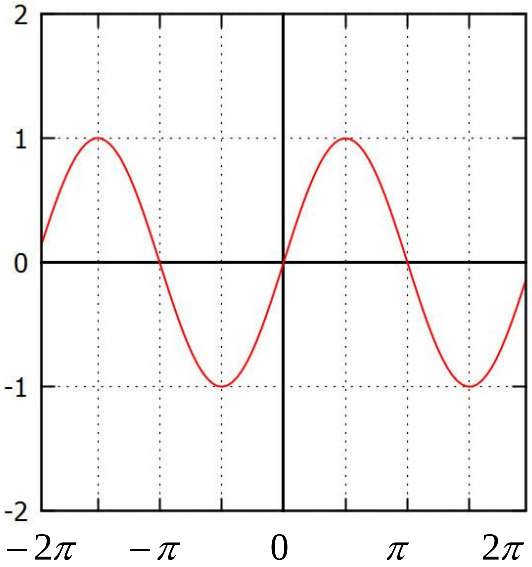
k > 1, expansion

0 < k < 1 compression

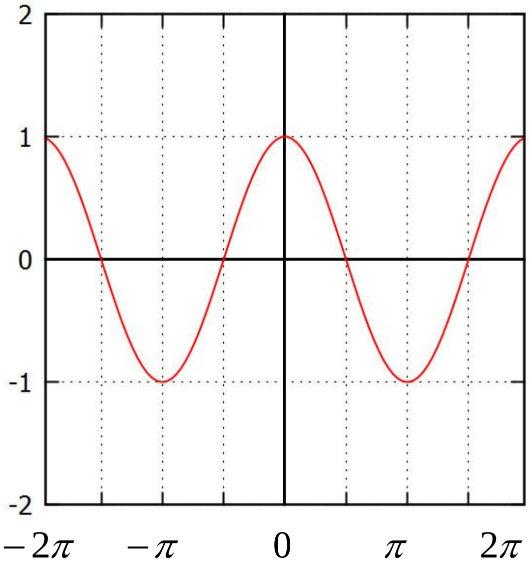
Standard Functions

You should be comfortable with sketching the following functions by hand:

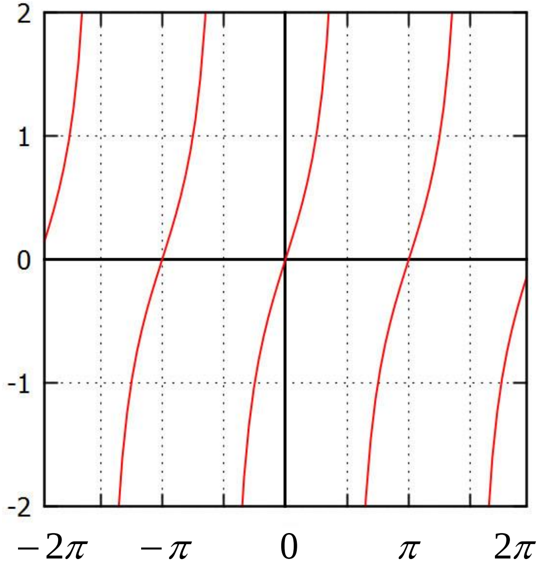
$f(x) = \sin(x)$



$f(x) = \cos(x)$



$f(x) = \tan(x)$

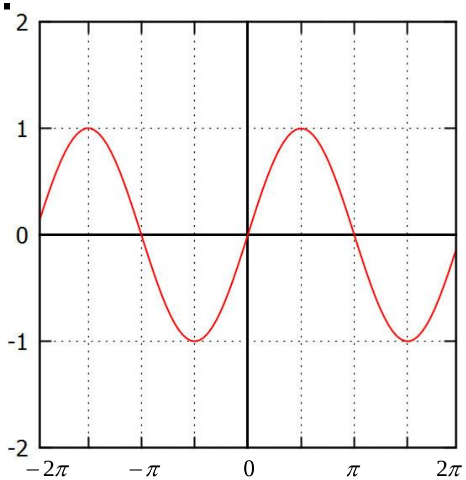


Transformations on Trigonometric Functions

The function $f(x) = \sin(x)$ is phase shifted (translated horizontally) by 2π.

Which graph shows this translation?

A.



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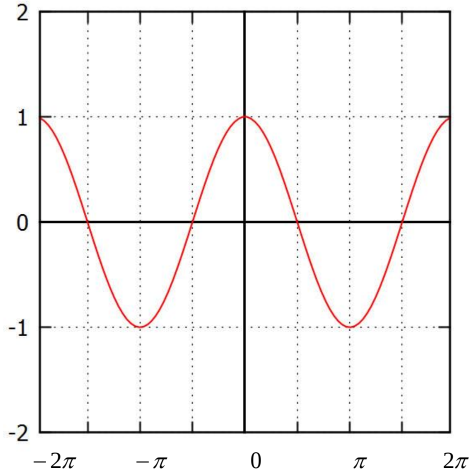
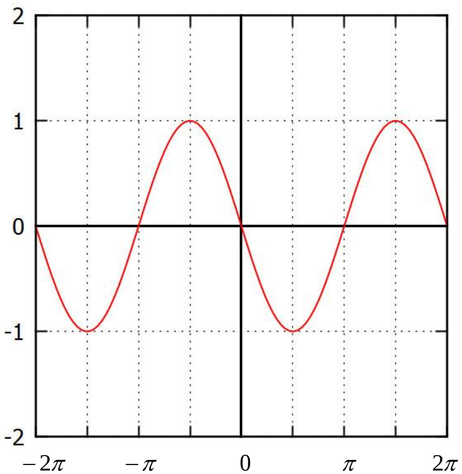
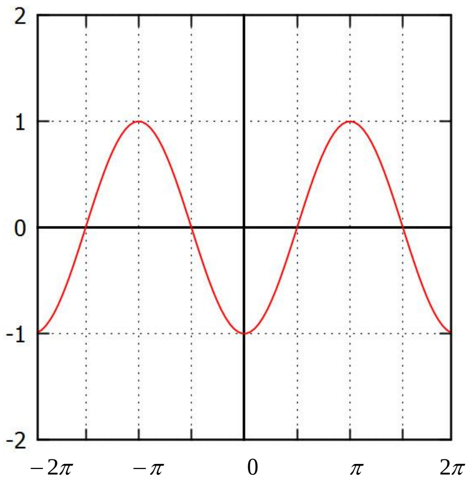


C.



D.

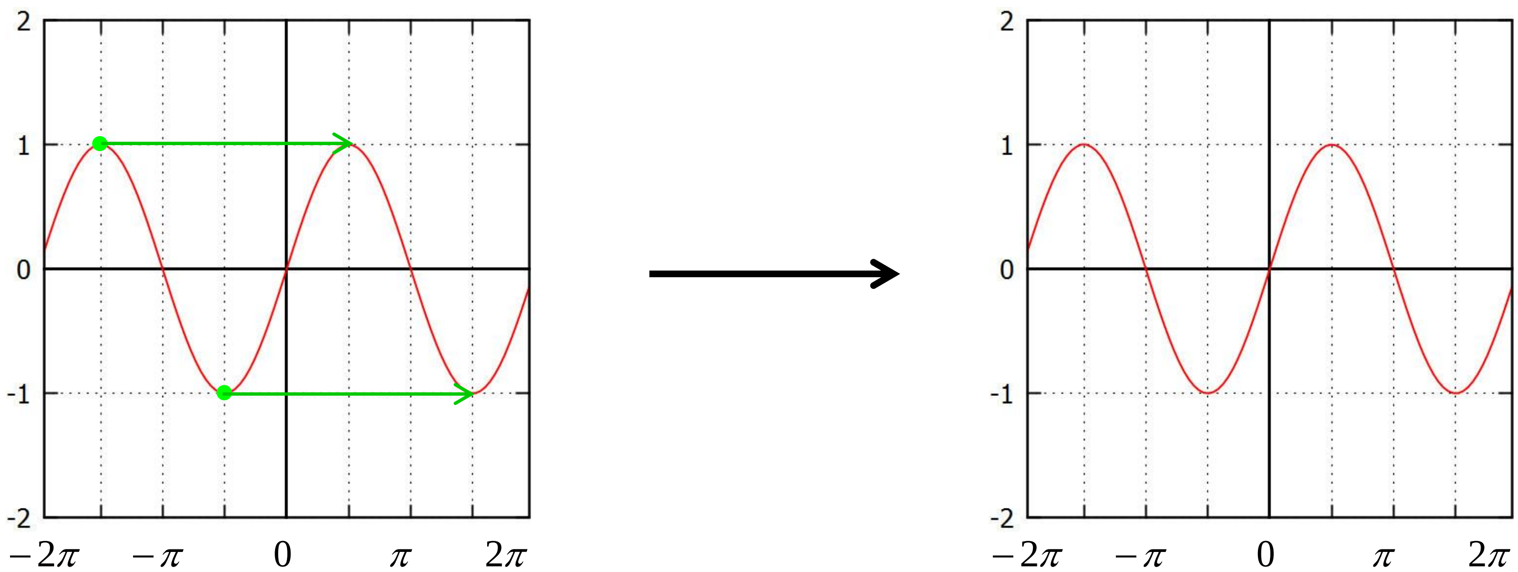


Solution

Answer: A

Justification: Since $\sin(x)$ is periodic with period 2π, shifting the sine curve by 2π left or right will not change the function.

$$\sin(x) = \sin(x - 2\pi) = \sin(x + 2\pi)$$



$f(x) = \sin(x)$ $\qquad$ $g(x) = \sin(x - 2\pi)$

Transformations on Trigonometric Functions II

The function $f(x)=\cos(x)$ is phase shifted by *k* units such that:

$$g(x)=\cos(x-k)=\sin(x)$$

Which of the following is a possible value of *k*?

A. $k=\frac{2\pi}{3}$

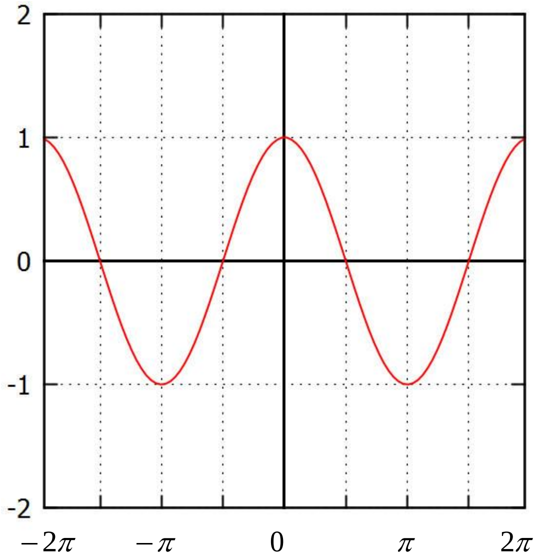
B. $k=\pi$

C. $k=\frac{\pi}{2}$

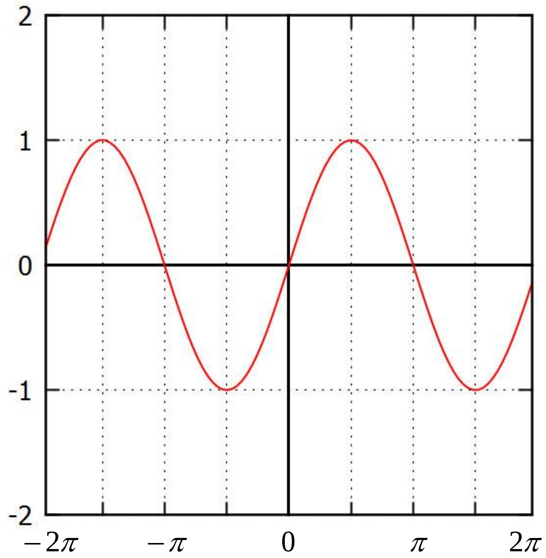
D. $k=-\frac{\pi}{2}$

E. $k=-\pi$

$f(x)=\cos(x)$



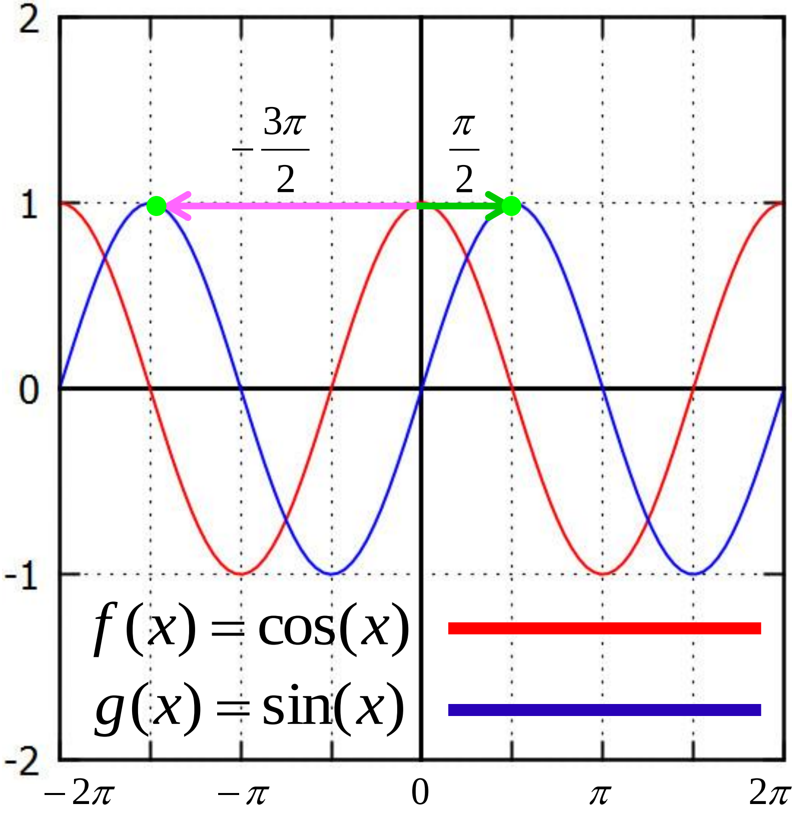
$g(x)=\sin(x)$



Solution

Answer: C

Justification: When translating right, the cosine graph must move $\frac{\pi}{2}$ units. When translating to the left, the cosine graph must move $\frac{3\pi}{2}$ units.



This explains the following trigonometric identities:

$$\sin(x) = \cos(x - \frac{\pi}{2}) = \cos(x + \frac{3\pi}{2})$$

The values of k are therefore:

$$k = \frac{\pi}{2} \quad \text{or} \quad k = -\frac{3\pi}{2}$$

Only answer C gives a possible value for k.

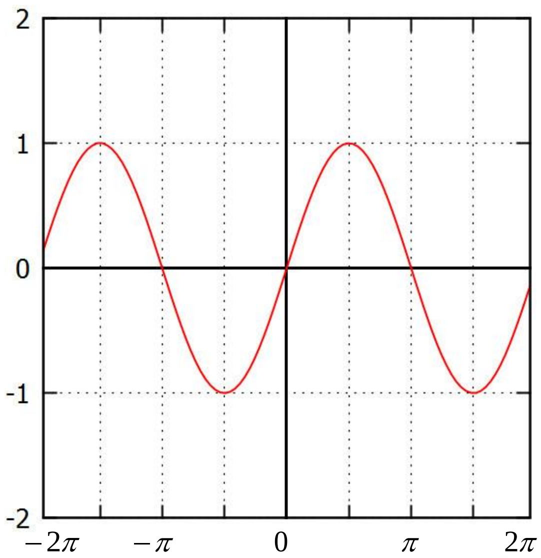
Transformations on Trigonometric Functions III

The function $f(x) = \sin(x)$ is phase shifted by *k* units such that:

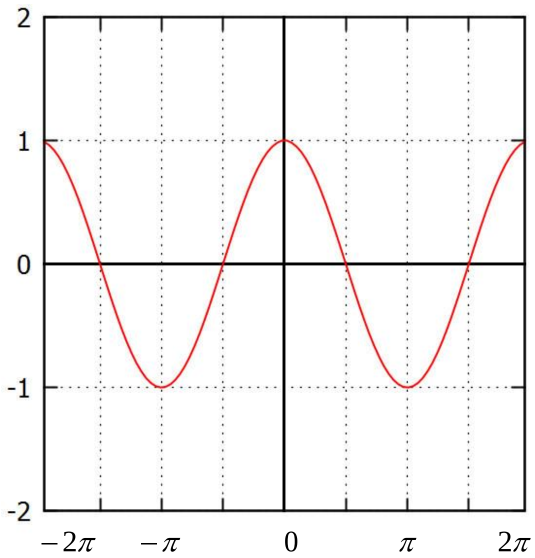
$$g(x) = \sin(x - k) = \cos(x)$$

Which of the following is a possible value of *k*?

$f(x) = \sin(x)$



$g(x) = \cos(x)$



A. $k = \dfrac{2\pi}{3}$

B. $k = \dfrac{\pi}{2}$

C. $k = -\dfrac{3\pi}{2}$

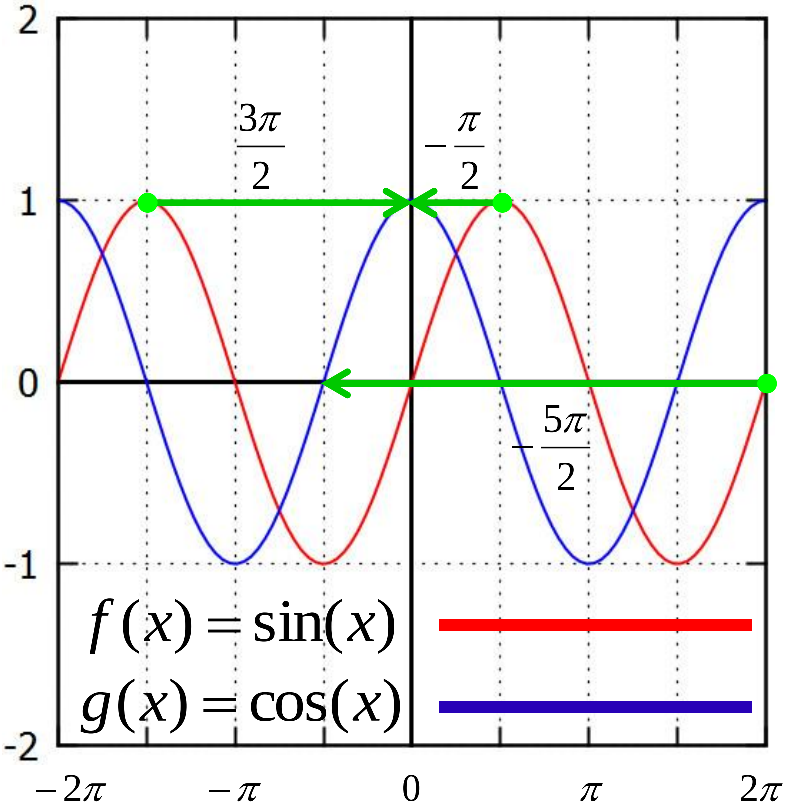
D. $k = -\dfrac{5\pi}{2}$

E. $k = -\dfrac{7\pi}{2}$

Solution

Answer: D

Justification: When translating right, the sine graph must move $\frac{3\pi}{2}$ units. When translating to the left, the sine graph must move $\frac{\pi}{2}$ units.



Neither of these values agree with the answers. Factors of 2π can be added or subtracted to these translations to reach the same outcome, since sine is periodic with 2π.

$$k = -\frac{\pi}{2} - 2\pi = -\frac{5\pi}{2}$$

$$\sin(x - k) = \cos(x)$$

$$\sin(x + \frac{5\pi}{2}) = \cos(x)$$

Transformations on Trigonometric Functions IV

The graph $g(x)$ shows the function $f(x) = \cos(x)$ after it has been phase shifted. Which of the following is true?

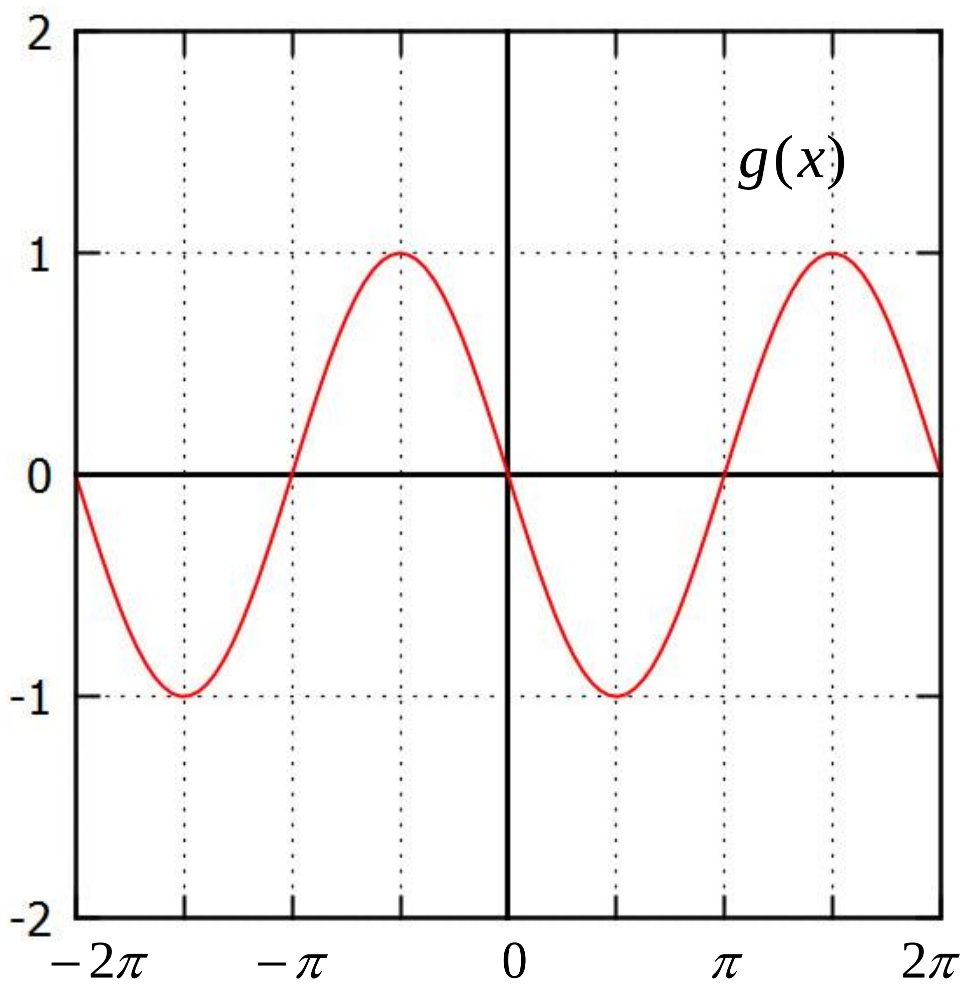
A. $g(x) = \cos(x + \frac{3\pi}{2})$

B. $g(x) = \cos(x + \pi)$

C. $g(x) = \cos(x - \frac{\pi}{2})$

D. $g(x) = \cos(x - \frac{3\pi}{2})$

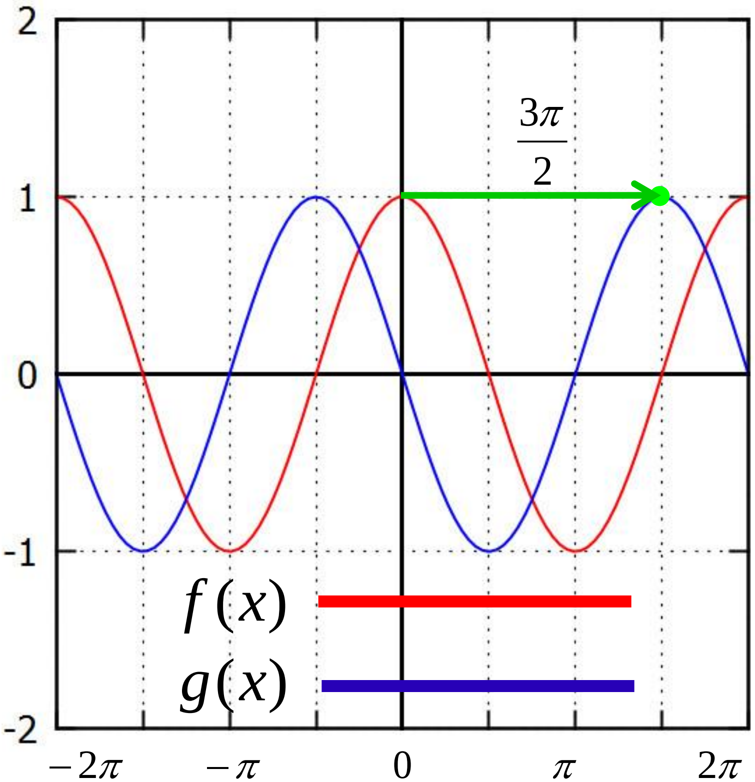
E. $g(x) = \cos(x - \frac{5\pi}{2})$



Solution

Answer: D

Justification: Find the first positive value where $g(x) = 1$. This point can be used to determine how much the cosine graph has been translated. Since $f(0) = \cos(0) = 1$, from the graph, we can see that the point (0, 1) has moved (right) to $(\frac{3\pi}{2}, 1)$.



The correct formula is therefore:

$$g(x) = \cos(x - \frac{3\pi}{2})$$

Note: If we instead shift left, an equivalent answer is:

$$g(x) = \cos(x + \frac{\pi}{2})$$

Transformations on Trigonometric Functions V

The graph $g(x)$ shows the function $f(x) = \tan(x)$ after it has been reflected. Across which axis has it been reflected?

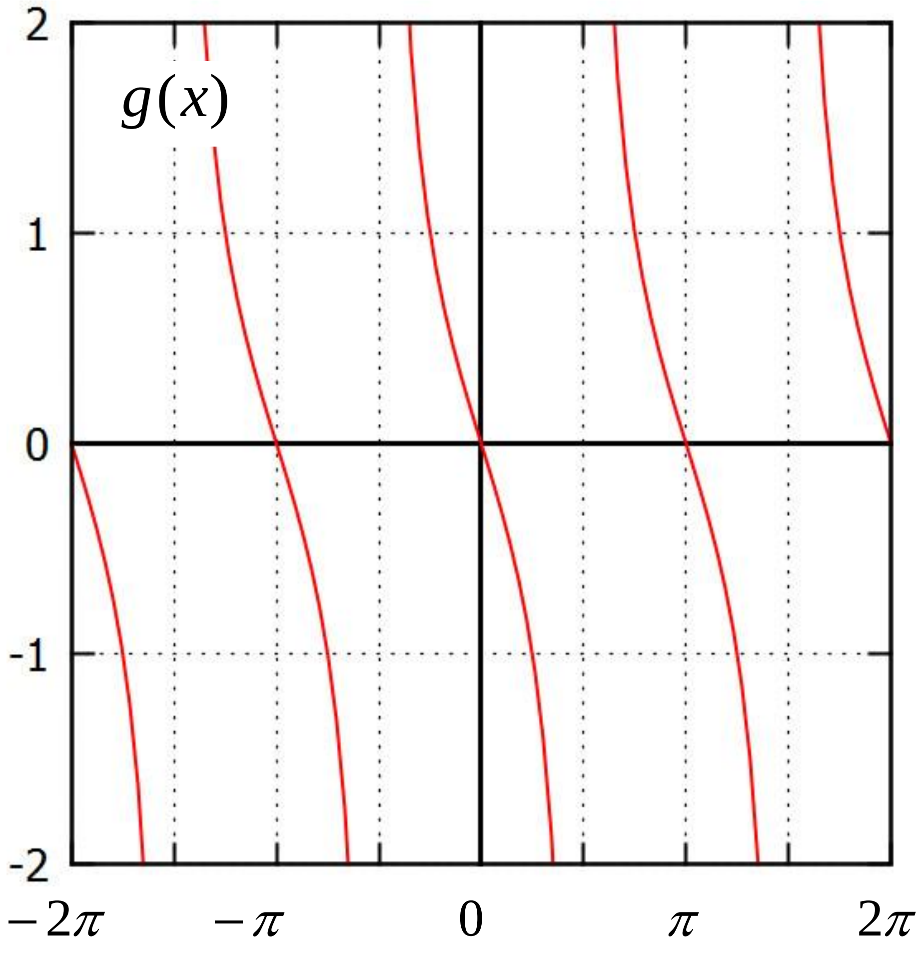
A. x-axis only

B. y-axis only

C. x-axis or y-axis

D. x-axis and y-axis

E. Neither x-axis or y-axis

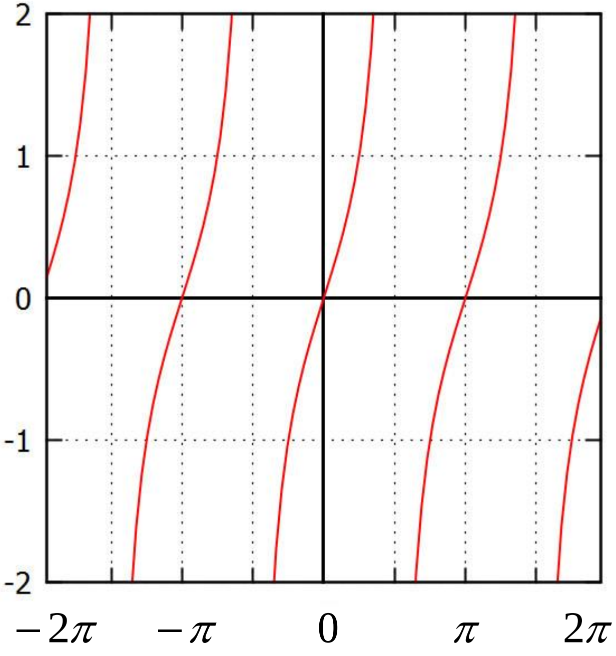


Solution

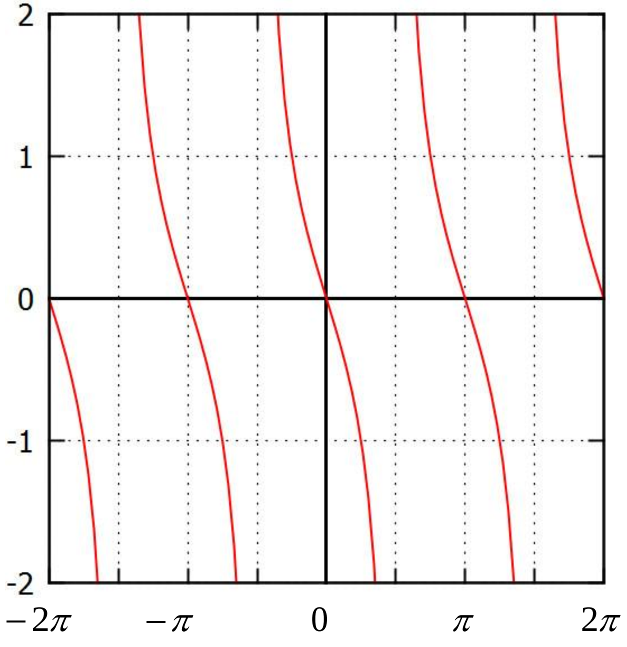
Answer: C

Justification: Since $f(x) = \tan(x)$ is an odd function, a reflection across the x-axis and a reflection across the y-axis are the same.

Recall that for odd functions $-f(x) = f(-x)$.



Reflection in x or y-axis

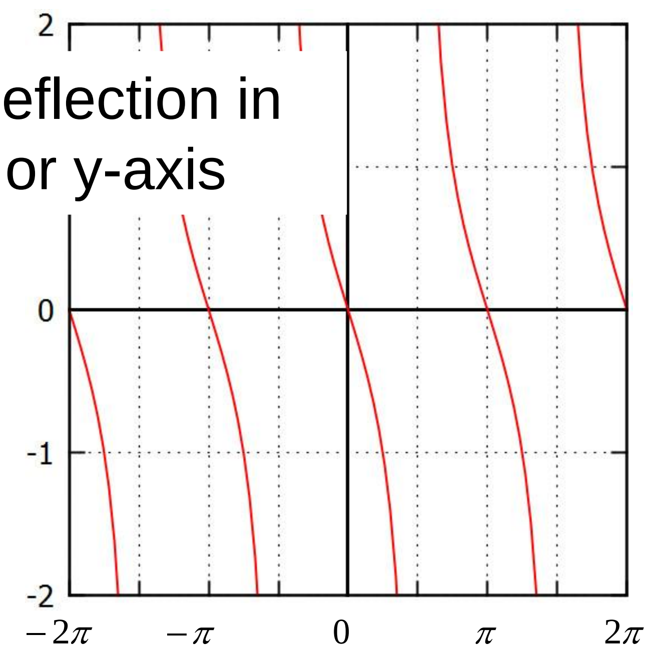


Solution Continued

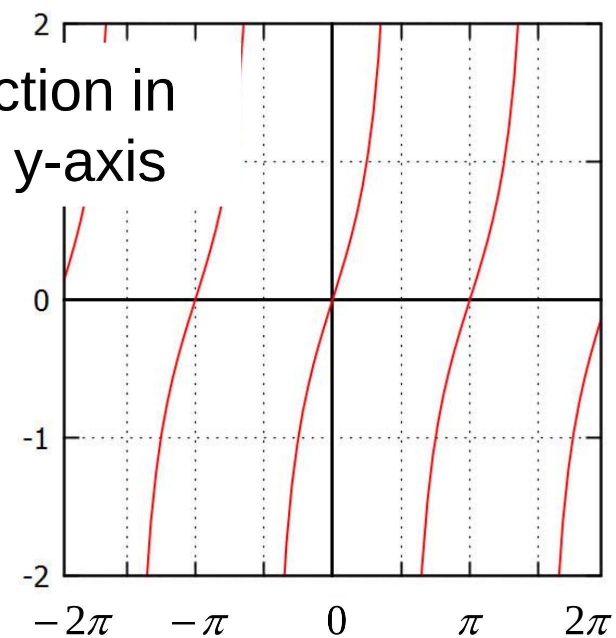
Answer: C

Justification: If we reflect $f(x) = \tan(x)$ in both the x-axis and y-axis, we would get the tangent function again. This is not the same as the graph $g(x)$.

$g(x)$



$f(x) = \tan(x) = -\tan(-x)$



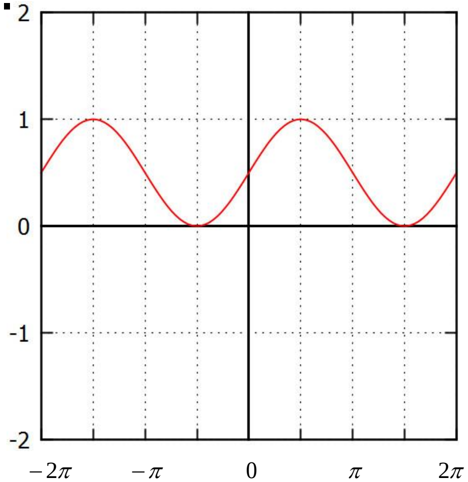
Transformations on Trigonometric Functions VI

$f(x) = \sin(x)$ has been displaced vertically such that

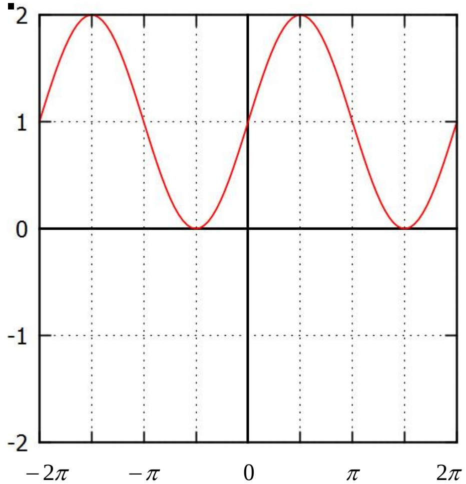
$$g(x) = \sin(x) + 1.$$

Which graph shows $g(x)$?

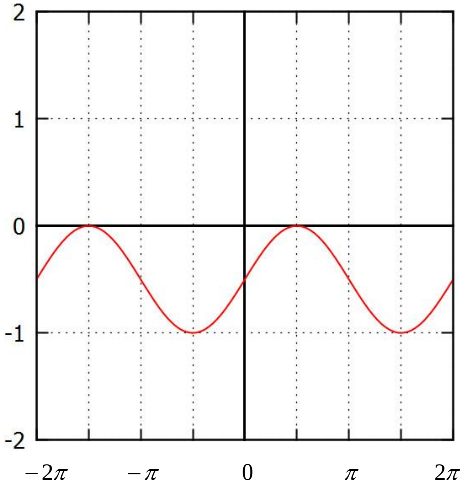
A.



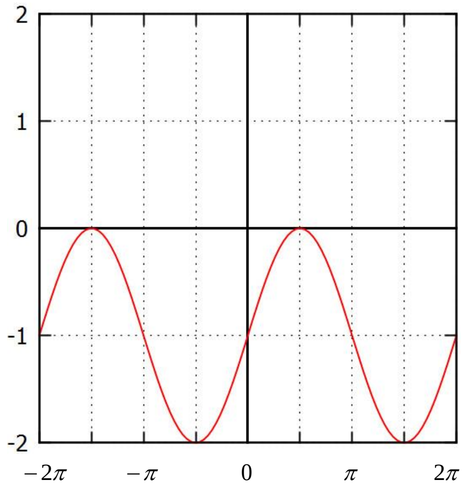
B.



C.



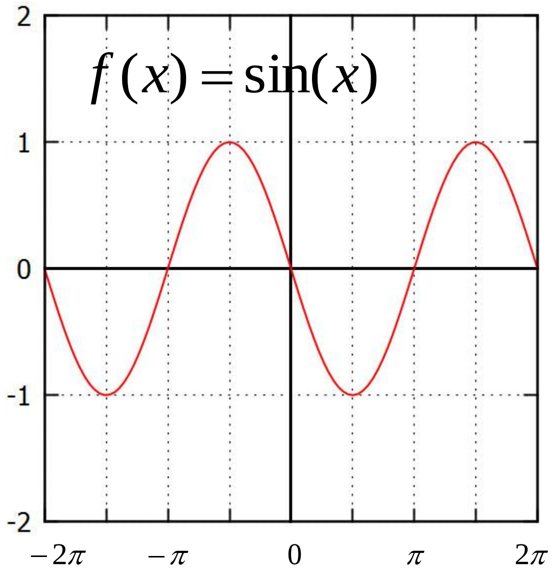
D.

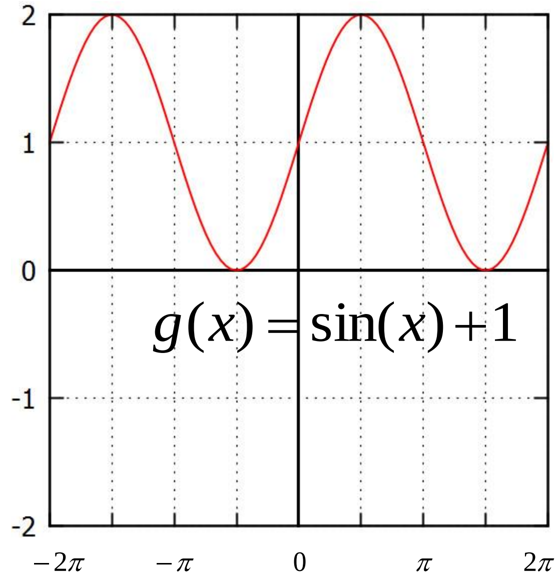


Solution

Answer: B

Justification: The transformation $g(x) = \sin(x) + 1$ shifts the graph vertically upwards by 1 unit. This eliminates answers C and D, which both represent downward shifts. Since $f(x) = \sin(x)$ spans between -1 and 1, we should expect g to span between 0 and 2.





$$-1 \le \sin(x) \le 1$$
$$-1+1 \le \sin(x)+1 \le 1+1$$
$$0 \le \sin(x)+1 \le 2$$

Transformations on Trigonometric Functions VII

The amplitude of a periodic function is half the difference between its maximum and minimum values.

What is the amplitude of

$$f(x) = 2\cos(x)\,?$$

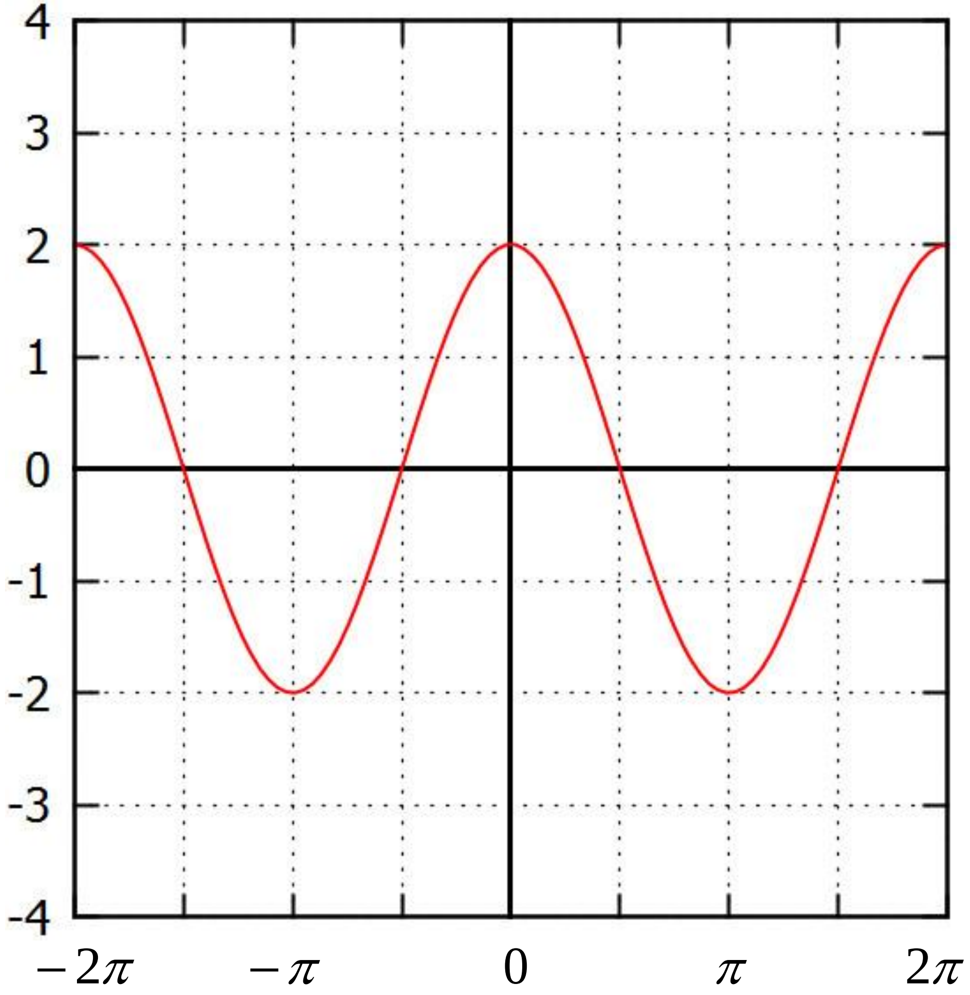
A. 8

B. 4

C. 2

D. 1

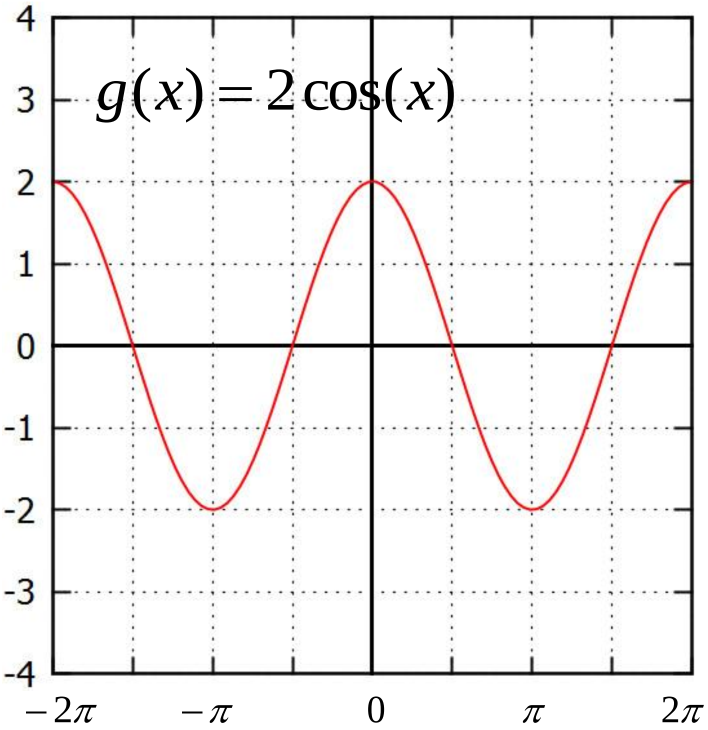
E. 0



Solution

Answer: C

Justification: From the graph of $g(x) = 2\cos(x)$, we can see the maximum value is 2 and the minimum value is -2.



Half the difference between the maximum and minimum is:

$$A = \frac{M - m}{2} \quad \text{where} \quad A = \text{amplitude}, \; M = \text{maximum}, \; m = \text{minimum}$$

$$= \frac{2 - (-2)}{2}$$

$$= 2$$

Transformations on Trigonometric Functions VIII

What is the amplitude of $f(x) = a \cdot \cos(x)$, where $a \neq 0$?

A. $2a$

B. a

C. 0

D. $|a|$

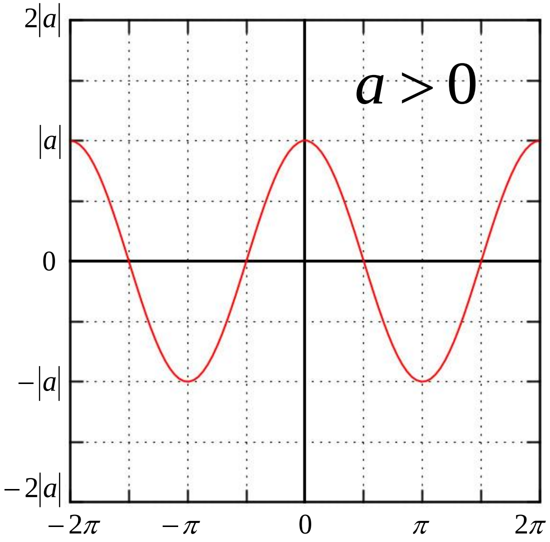
E. Cannot be determiend

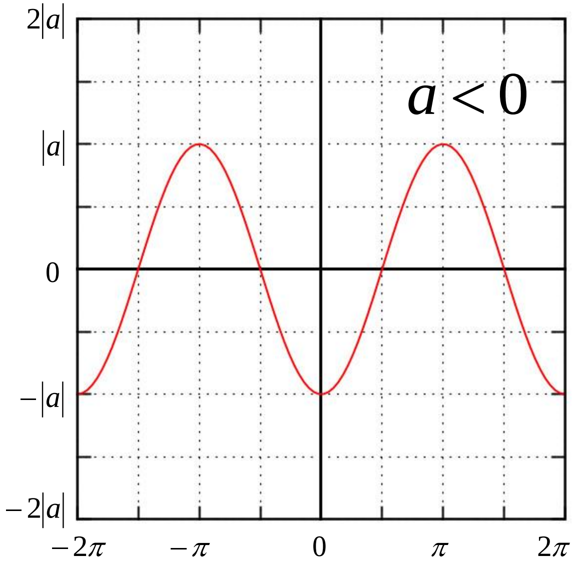
Solution

Answer: D

Justification: The amplitude of $f(x) = a \cdot \cos(x)$ is calculated as shown:

$$A = \frac{M - m}{2}$$
$$= \frac{|a| - (-|a|)}{2}$$
$$= |a|$$





The two graphs above show that the maximum and minimum values of $f(x) = a \cdot \cos(x)$ are $|a|$ and $-|a|$ respectively. The amplitude cannot be negative.

Transformations on Trigonometric Functions IX

The function $f(x) = \cos(x)$ is vertically expanded and displaced so that $g(x) = p \cdot \cos(x) + q$. If

$$-6 \le g(x) \le 2,$$

what are the values of *p* and *q*?

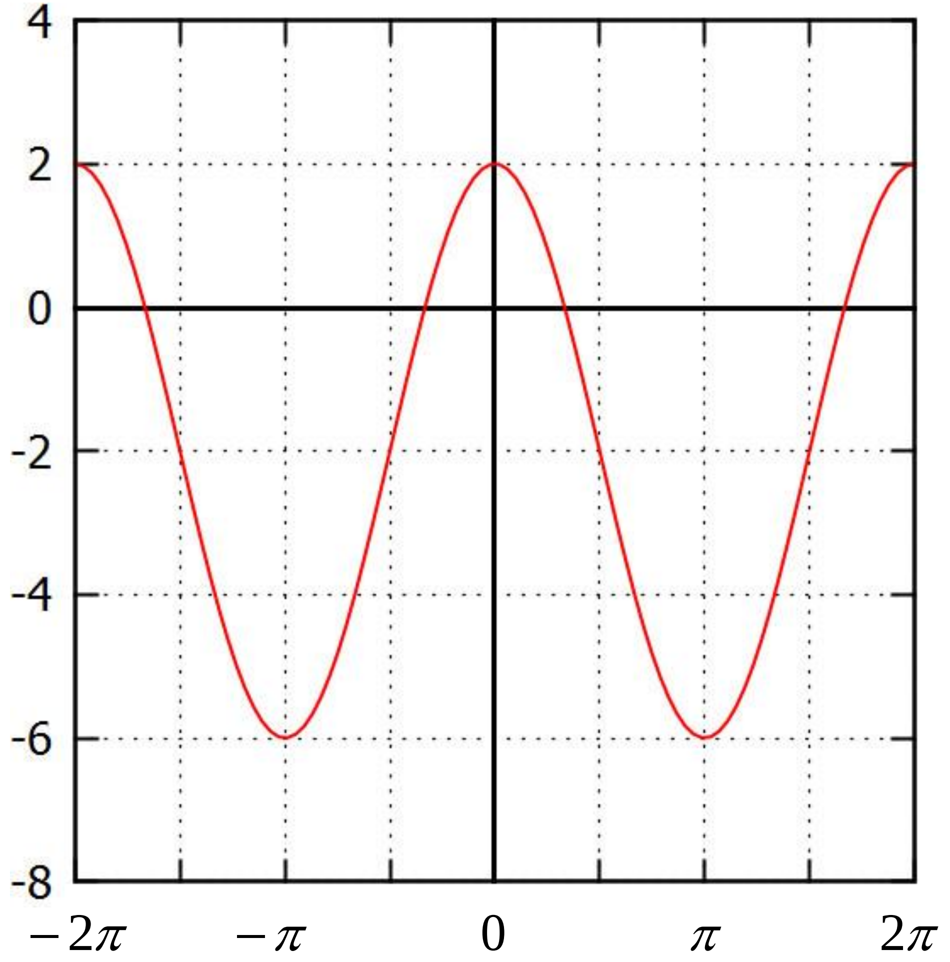
A. $p = 4, \quad q = -2$

B. $p = 4, \quad q = 2$

C. $p = 8, \quad q = -4$

D. $p = 8, \quad q = -2$

E. $p = 8, \quad q = 2$



Solution

Answer: A

Justification: We are given that $-6 \le g(x) \le 2$, so its maximum value is 2 and its minimum is -6. We can calculate the amplitude:

$$A = \frac{M-m}{2} = \frac{2-(-6)}{2} = 4$$

Vertical displacement does not change the shape of the graph, therefore it does not impact amplitude.

We can determine that $4\cos(x)$ spans between -4 and 4 using what we learned from the previous question. In order to change the max value from 4 to 2 (or the min from -4 to -6), we must shift the function down by 2 units:

$$-4 \le 4\cos(x) \le 4$$

Translate 2 units down $-6 \le 4\cos(x) - 2 \le 2$ $\Rightarrow$

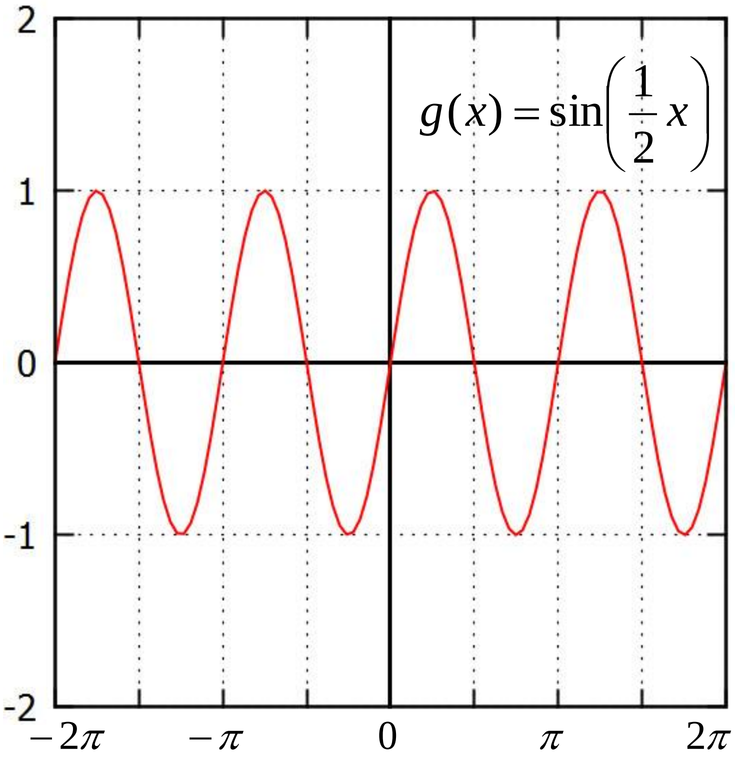
$$g(x) = 4 \cdot \cos(x) - 2$$

$$p = 4, \quad q = -2$$

Transformations on Trigonometric Functions X

What is the period of the function $g(x) = \sin\left(\frac{1}{2}x\right)$?

Press for hint



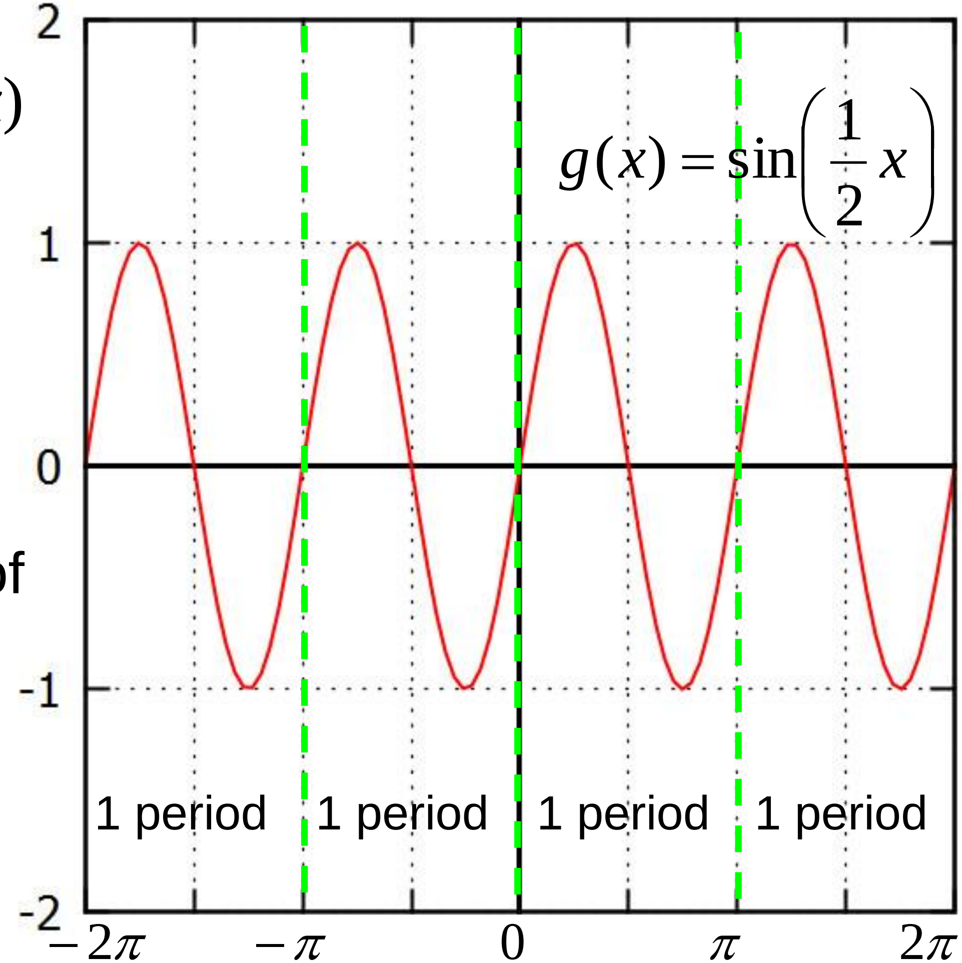
A. 4
B. 2
C. 2π
D. π
E. $\frac{\pi}{2}$

Solution

Answer: D

Justification: The function $g(x)$ shows the graph $\sin(x)$ after it has been horizontally compressed by a factor of 0.5.

We should expect the period of $g(x)$ to be compressed by the same factor. Since the period of $\sin(x)$ is 2π, the period of $\sin(0.5x)$ is π.



Transformations on Trigonometric Functions XI

What is the period of the function $g(x) = a \cdot \tan(bx - c) + d,\ a \neq 0$?

A. $2b\pi$

B. $b\pi$

C. b

D. $\dfrac{b\pi}{2}$

E. $\dfrac{\pi}{b}$

The period of the tangent function $f(x) = \tan(x)$ is π.

Press for hint

Solution

Answer: E

Justification: Recall that horizontal stretches by a factor of k results in substituting x with x/k.

Since $g(x) = a \cdot \tan(bx - c) + d$ has been horizontally stretched by a factor of $1/b$ and the period of $\tan(x)$ is π, the period of g is π/b.

Notice that the vertical stretch by the factor of a does not affect the period of the function. The vertical displacement by d units and phase shift by c units do not change the shape of a function, so they also do not affect the period of the function. The period of the sine, cosine, and tangent functions are only dependant on the horizontal stretch, b.

Transformations on Trigonometric Functions XII

The graph of $g(x) = \sin(bx)$ is shown to the right. What is the value of *b*?

Pay attention to the values on the x-axis.

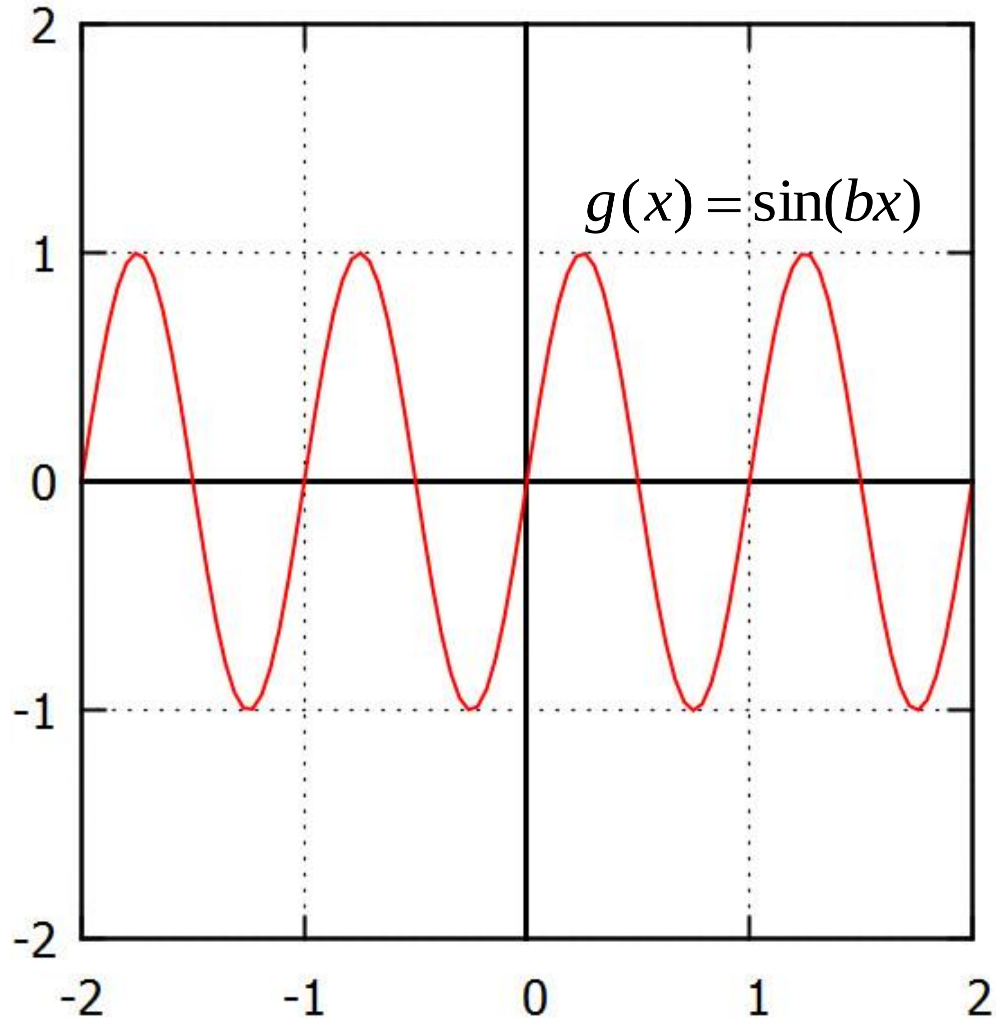
A. $b = 2\pi$

B. $b = \pi$

C. $b = 2$

D. $b = \dfrac{2}{\pi}$

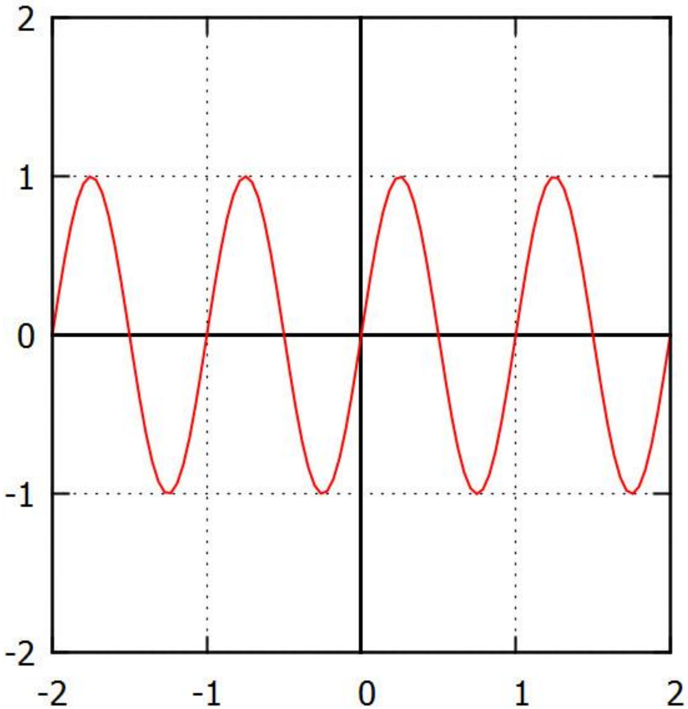
E. $b = \dfrac{1}{2\pi}$



Solution

Answer: A

Justification: The period of the function shown in the graph is 1. The period of $g(x) = \sin(bx)$ is $2\pi/b$. (Review the solution to the previous question, except using $\sin(x)$ rather than $\tan(x)$.



We can solve for b to find by solving:

$$\frac{2\pi}{b} = 1$$

$$b = 2\pi$$

The graph has been horizontally compressed by a factor of $1/2\pi$.

$$g(x) = \sin(2\pi x)$$

Transformations on Trigonometric Functions XIII

The function $g(x) = a \cdot \tan(0.5x)$ is shown to the right. What is the approximate value of *a*?

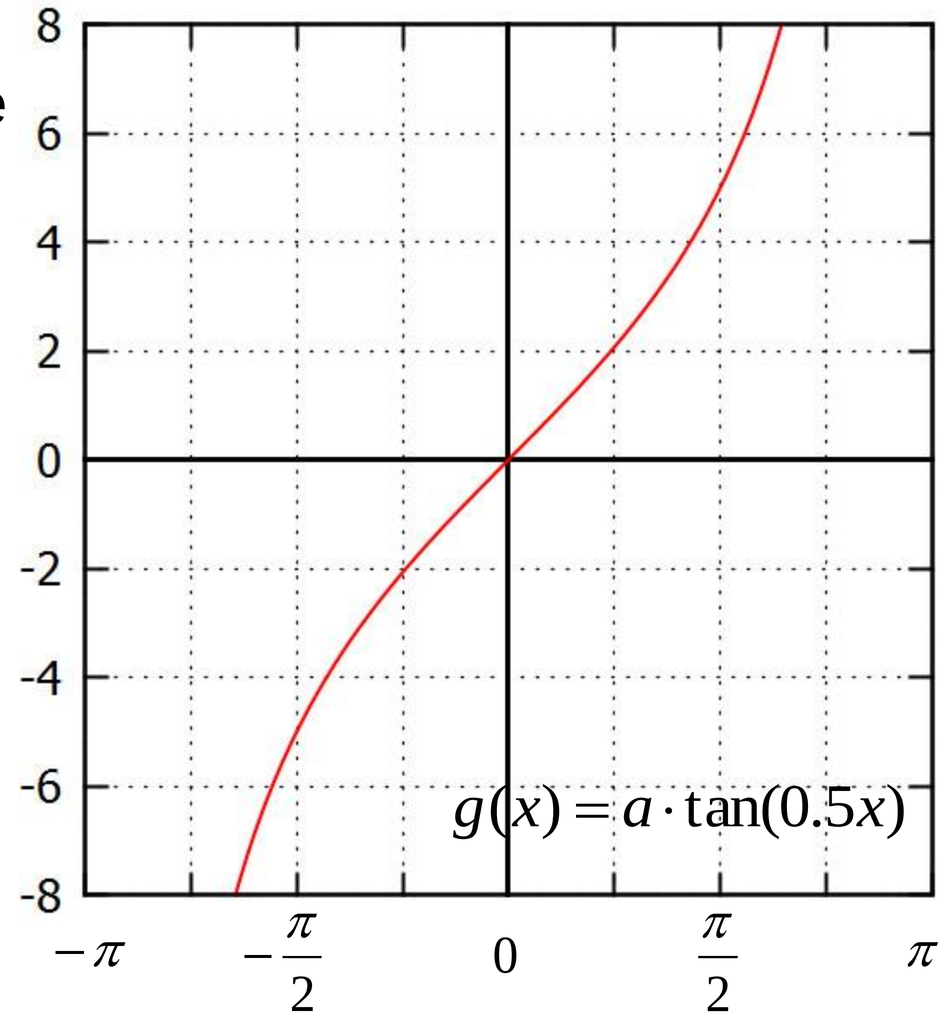
A. $8 < a < 10$

B. $6 < a < 8$

C. $4 < a < 6$

D. $2 < a < 4$

E. $0 < a < 2$



Press for hint

There exists a value *x* = *p* where $g(p) = a$.

$$\tan\left(\frac{\pi}{4}\right) = 1$$

$$a \cdot \tan\left(\frac{\pi}{4}\right) = a$$

Solution

Answer: C

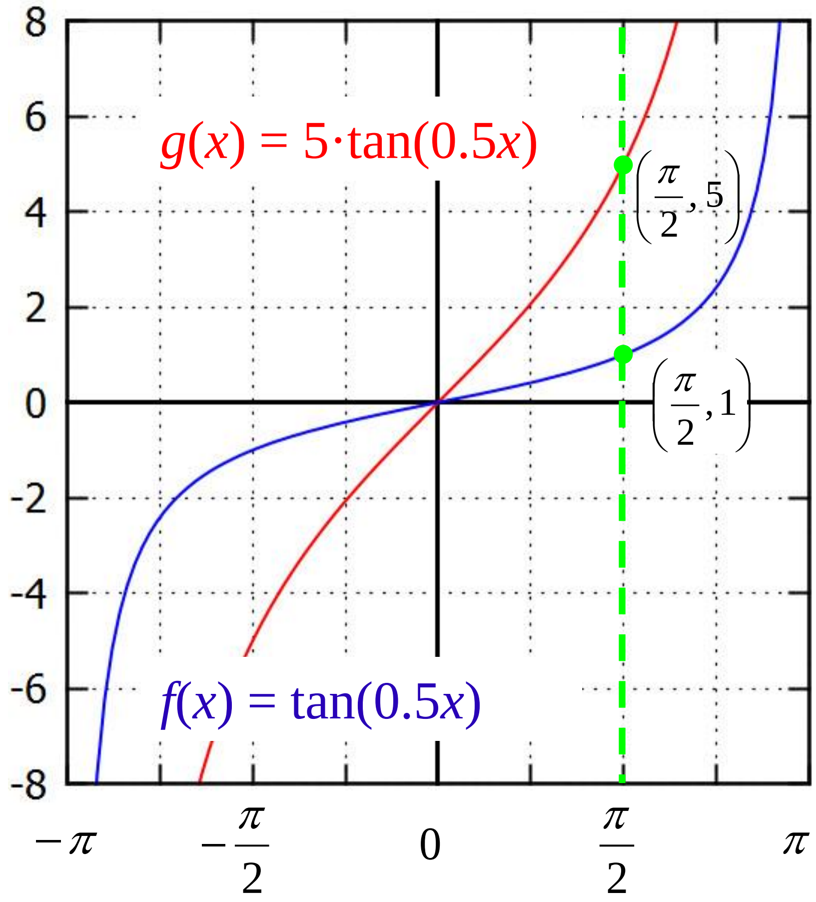
Justification: Recall that $\tan\left(\frac{\pi}{4}\right)=1$

A tangent function that has been horizontally expanded by 2 will equal one at $\pi/2$, rather than $\pi/4$.

$g(x)=a\cdot\tan\left(\frac{x}{2}\right)$ Let $x=\frac{\pi}{2}$

$$g\left(\frac{\pi}{2}\right)=a\cdot\underbrace{\tan\left(\frac{1}{2}\frac{\pi}{2}\right)}_{1}=a$$

From the graph, $g\left(\frac{\pi}{2}\right)=a=5$.



Points whose y-values are 1 before being vertically stretched reveal the expansion or compression factor.