a place of mind

FACULTY OF EDUCATION

Department of
Curriculum and Pedagogy

Mathematics

Numbers: Exponents

Science and Mathematics
Education Research Group

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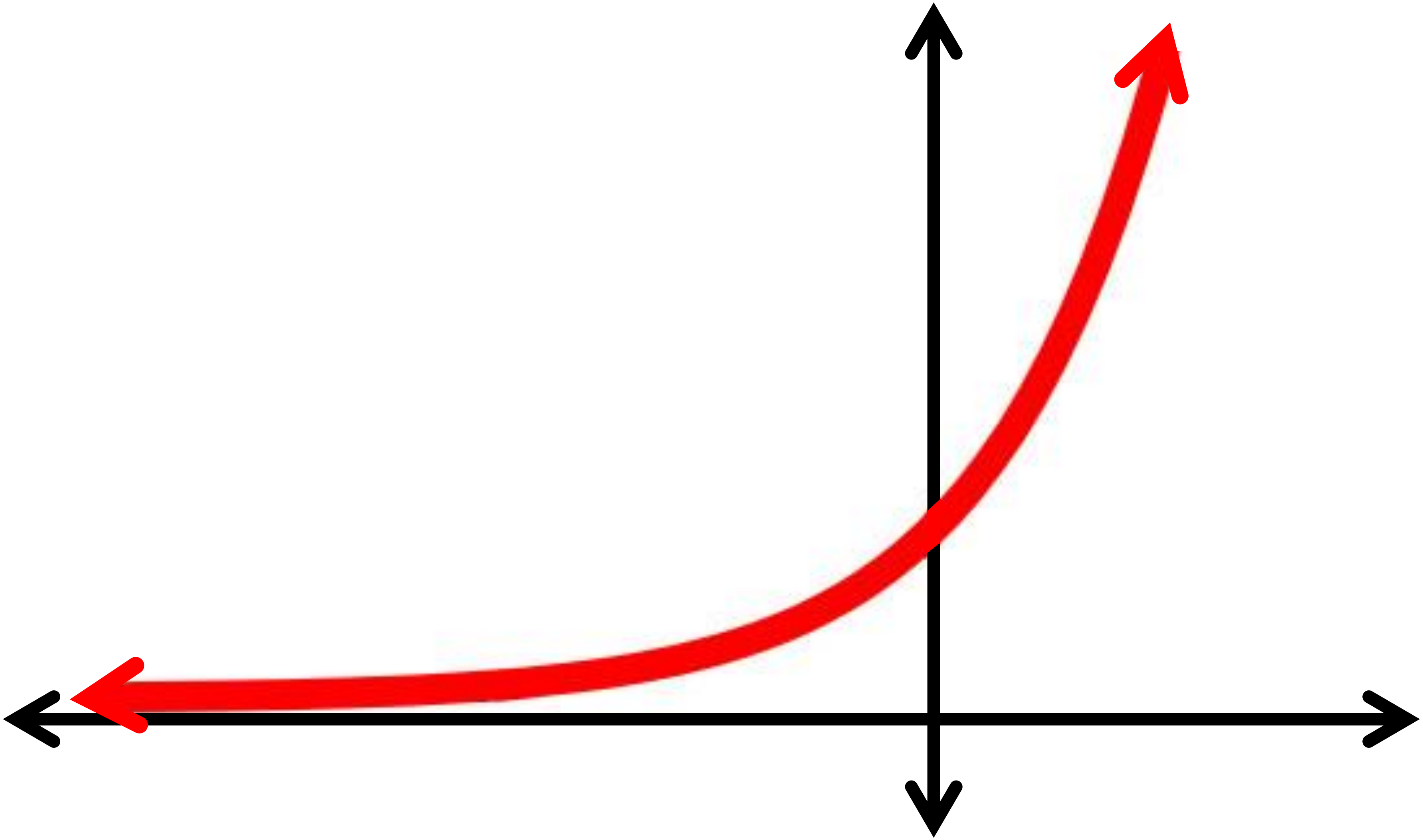
Properties of Exponents



Review of Exponent Laws

Let a and b be positive real numbers. Let x and y be real numbers.

$$a^0 = 1$$

$$a^x \cdot a^y = a^{x+y}$$

$$\frac{a^x}{a^y} = a^{x-y}$$

$$\left(a^x\right)^y = a^{xy}$$

$$a^{-x} = \frac{1}{a^x}$$

$$\frac{a^x}{b^x} = \left(\frac{a}{b}\right)^x$$

$$\left(\frac{a}{b}\right)^{-x} = \left(\frac{b}{a}\right)^x$$

$$a^x = a^y \quad \text{if and only if } x = y$$

Exponent Laws I

How can the following expression be simplified using exponents?

$$3 \times 3 \times 3 + 3 \times 3$$

A. 3^5

B. $3^3 + 3^2$

C. $(3+2)^3$

D. $3^3 + 2^3$

E. Either A or B

$$a^x + a^y \neq a^{x+y}$$

Press for hint

Solution

Answer: B

Justification: Group the expression into two separate powers of 3:

$$\underbrace{3\times3\times3}_{3^3}+\underbrace{3\times3}_{3^2}$$

This gives the final expression $3^3 + 3^2$. This cannot be simplified any further unless the powers of three are calculated.

Answer A (3^5) is incorrect since $3^3 + 3^2 \neq 3^5$. Notice that we only add exponents when two powers with the same base are multiplied together, not added:

$$a^x \cdot a^y = a^{x+y} \qquad\qquad a^x + a^y \neq a^{x+y}$$

Solution Continued

The values we are comparing in this question are small, so we can calculate the final values of each expression:

$$3\times3\times3+3\times3=27+9=36$$

A. $3^5=243$

B. $3^3+3^2=27+9=36$

C. $(3+2)^3=5^3=125$

D. $3^3+2^3=27+8=35$

Only answer B matches the value of the expression in the question. The other answers give a different final value, so the expressions are not equivalent.

Exponent Laws II

Simplify the following expression:

$$\frac{7^5 + 7^2}{7^2}$$

A. $7^3 + 1$

B. $7^5 + 1$

C. 7^5

D. $7^3 + 7^2$

E. Cannotbesimplifie

$$\frac{7^5 + 7^2}{7^2} = \frac{7^5}{7^2} + \frac{7^2}{7^2}$$

Press for hint

Solution

Answer: A

Justification: This expression can be simplified in several ways:

Factor out 7^2 from the numerator:

$$\frac{7^5+7^2}{7^2}=\frac{\cancel{7^2}(7^3+7^0)}{\cancel{7^2}}$$

$$=7^3+7^0$$

$$=7^3+1$$

Split the fraction into the sum of two fractions:

$$\frac{7^5+7^2}{7^2}=\frac{7^5}{7^2}+\frac{7^2}{7^2}$$

$$=7^{5-2}+7^{2-2}$$

$$=7^3+7^0$$

$$=7^3+1$$

Exponent Laws III

Simplify the following expression: $\dfrac{12^{10}}{3^{10}4^{12}}$

A. 16

B. 12

C. $\dfrac{1}{4}$

D. $\dfrac{1}{12}$

E. $\dfrac{1}{16}$

$$(ab)^x = a^x b^x$$

Press for hint

Solution

Answer: E

Justification: Write 12^{10} as a product of powers with base 3 and 4:

$$\frac{12^{10}}{3^{10}4^{12}} = \frac{(3\cdot 4)^{10}}{3^{10}4^{12}} = \frac{\cancel{3}^{10}4^{10}}{\cancel{3}^{10}4^{12}} = \frac{1}{4^{12-10}} = \frac{1}{16}$$

Alternatively, you can write the denominator as a power with base 12:

$$\frac{12^{10}}{3^{10}4^{12}} = \frac{12^{10}}{3^{10}4^{10}4^{2}} = \frac{12^{10}}{(3\cdot 4)^{10}4^{2}} = \frac{\cancel{12}^{10}}{\cancel{12}^{10}4^{2}} = \frac{1}{4^{2}} = \frac{1}{16}$$

Exponent Laws IV

Simplify the following expression: $$\frac{4^2}{2^8 + 2^9}$$

A. $\frac{1}{2^6 + 2^7}$

B. $\frac{1}{2^4 + 2^5}$

C. $\frac{1}{2^4 + 2^9}$

D. $\frac{1}{2^9}$

E. Cannot be simplifie

Solution

Answer: B

Justification: The power of 4 can be rewritten as a power with a base of 2:

$$\frac{4^2}{2^8+2^9}=\frac{(2^2)^2}{2^8+2^9}=\frac{2^{2\cdot 2}}{2^8+2^9}=\frac{2^4}{2^8+2^9}$$

Factor out a power of 2 from the denominator to cancel with the numerator:

$$\frac{2^4}{2^8+2^9}=\frac{2^4}{2^4\cdot 2^{8-4}+2^4\cdot 2^{9-4}}=\frac{\cancel{2^4}}{\cancel{2^4}(2^4+2^5)}=\frac{1}{2^4+2^5}$$

Exponent Laws V

Write the following as a single power with base 4: $$\frac{4^{17}}{4^0 \cdot 4^{17} + 4^0}$$

A. 4^{17}

B. $4^{\frac{1}{2}}$

C. 4^0

D. 4^{-1}

E. Cannot be written as a single power of 4

Solution

Answer: E

Justification: This expression cannot be simplified any further.

$$\frac{4^{17}}{4^{17}+4^{0}} = \frac{4^{17}}{4^{17}+1}$$

Common errors include:

1. Incorrectly adding exponents

$$\frac{4^{17}}{4^{17}+4^{0}} \neq \frac{4^{17}}{4^{17+0}} = 4^{0}$$

2. $4^0 = 1$, not 0

$$\frac{4^{17}}{4^{17}+4^{0}} \neq \frac{4^{17}}{4^{17}+0} = 4^{0}$$

3. Splitting the denominator

$$\frac{4^{17}}{4^{17}+4^{0}} \neq \frac{4^{17}}{4^{17}} + \frac{4^{17}}{4^{0}}$$

Exponent Laws VI

Write the following as a single power of 2:

$$\left(\frac{2}{0.5}\right)^5 (0.5)^4 (2)^{-5}$$

A. 2^2

B. 2^1

C. 2^0

D. 2^{-1}

E. Cannot be written as a single power of 2

Solution

Answer: B

Justification: This expression can be simplified in many ways because all the terms can be expressed as a power of 2. Two possible solutions are shown below.

Cancel terms where possible and collect like terms:

$$\left(\frac{2}{0.5}\right)^5 (0.5)^4 (2)^{-5} = \frac{2^5}{0.5^5} \cdot 0.5^4 \cdot \frac{1}{2^5} = \frac{1}{0.5^{5-4}} = 2^1$$

Express all terms as a power of 2:

$$\left(\frac{2}{0.5}\right)^5 (0.5)^4 (2)^{-5} = (2 \cdot 2)^5 \cdot \left(\frac{1}{2}\right)^4 \cdot (2)^{-5} = 2^5 \cdot 2^5 \cdot 2^{-4} \cdot 2^{-5} = 2^{5+5-4-5} = 2^1$$

Exponent Laws VII

Which of the following powers is the largest?

$$2^5 \qquad 3^4 \qquad 4^3 \qquad 5^2 \qquad 6^{-5}$$

A. 2^5

B. 3^4

C. 4^3

D. 5^2

E. 6^{-5}

Solution

Answer: B

Justification: There are generally no rules when comparing powers with different bases.

$$6^{-5} < 5^2 < 2^5 < 4^3 < 3^4$$

$$2^5 = 32$$

$$3^4 = 81$$

$$4^3 = 64$$

$$5^2 = 25$$

$$6^{-5} = \frac{1}{6^5} = \frac{1}{7776}$$

Note: Large exponents tend to have more impact on the size of a number than large bases.

$$2^{100} >> 100^2$$

Exponent Laws VIII

How many of the following terms are less than 0?

$$(-3)^2 \qquad 3^{-2} \qquad -3^{-2} \qquad -3^2$$

A. 0

B. 1

C. 2

D. 3

E. 4

Solution

Answer: C

Justification: Simplify each term separately.

Be careful when dealing with negatives on exponents. Although every expression has a negative sign, only -3^2 and -3^{-2} are negative. Think about order of operations: brackets come before exponents, which come before multiplication.

$$(-3)^2 = 9$$

$$3^{-2} = \frac{1}{9}$$

$$-3^{-2} = -\frac{1}{9}$$

$$-3^2 = -9$$

$$-3^2 < -3^{-2} < 0 < 3^{-2} < (-3)^2$$

$$-9 < -\frac{1}{9} < 0 < \frac{1}{9} < 9$$

Exponent Laws IX

How many of the following are less than 1?

$$(0.5)^2, \quad 2^{-2}, \quad (-0.5)^{-2}, \quad (-0.5)^2, \quad -(0.5)^{-2}$$

A. 1
B. 2
C. 3
D. 4
E. 5

Solution

Answer: D

Justification: Simplify each term separately to find the terms less than 1.

$$(0.5)^2 = \frac{1}{2^2} = \frac{1}{4}$$ Less than 1

$$2^{-2} = \frac{1}{2^2} = \frac{1}{4}$$ Less than 1

$$(-0.5)^{-2} = (-2)^2 = 4$$ Greater than 1

$$(-0.5)^2 = \frac{1}{(-2)^2} = -\frac{1}{4}$$ Less than 1

$$-(0.5)^{-2} = -2^2 = -4$$ Less than 1

Exponent Laws X

Consider adding 3 until you obtain the value 3^{33} as shown below:

$$\underbrace{3+3+3...+3}_{\text{n terms}} = 3^{33}$$

How many terms are there in the summation?

A. $n = 3^{33} - 1$

B. $n = 3^{33} - 3$

C. $n = 3^{32}$

D. $n = 3^{11}$

E. $n = 33$

Solution

Answer: C

Justification: The summation is equal to 3*n*.

$$\underbrace{3+3+3...+3}_{n \text{ terms}}=3^{33}$$

$$3n=3^{33}$$

It becomes straightforward to solve for *n* after this step.

$$3n=3^{33}$$

$$n=\frac{3^{33}}{3^{1}}$$

$$n=3^{32}$$

Exponent Laws XI

Let *p* and *q* be positive integers. If $p > q$, which of the following are always true?

A. $2^{-p} > 2^{-q}$

B. $2^{-p} \geq 2^{-q}$

C. $2^{-p} = 2^{-q}$

D. $2^{-p} < 2^{-q}$

E. $2^{-p} \leq 2^{-q}$

Solution

Answer: D

Justification: Rewrite the two expressions using positive exponents of p and q:

$$2^{-p} = \frac{1}{2^p}, \quad 2^{-q} = \frac{1}{2^q}$$

It is now much easier to compare the two expressions.

$$\frac{1}{2^p} < \frac{1}{2^q} \quad \text{since} \quad 2^p > 2^q$$

Remember that dividing by a larger denominator gives a smaller result.

Exponent Laws XII

Let p and q be positive integers and $p > q$. If $b > 0$, find all values of b such that

$$b^{-p} \geq b^{-q}$$

is always true.

A. $b = 1$

B. $0 < b < 1$

C. $0 < b \leq 1$

D. $b > 1$

E. $b \geq 1$

Solution

Answer: C

Justification: Rewrite the inequality using positive exponents of *p* and *q*:

$$\frac{1}{b^p} \geq \frac{1}{b^q}$$

The LHS is larger than the RHS only if $b^p < b^q$, since numbers with smaller denominators are larger. Therefore *b* must be between 0 and 1 to make $b^p < b^q$ (since $p > q$). For example, $0.5^2 < 0.5^3$.

In order to choose between answers B and C, consider when $b = 1$.

$$\frac{1}{1^p} = \frac{1}{1^q}$$ since 1 to any power is still 1

Since we have to include the equality case, the answer is C:

$$0 < b \leq 1$$

Exponent Laws XIII

Solve for c.

$$\left(2^{\left(2^{x}\right)}\right)^{c} = 2^{\left(2^{x+1}\right)}$$

A. $c = \frac{1}{2}$

B. $c = 2$

C. $c = 4$

D. $c = 8$

E. $c = 2^{x}$

Solution

Answer: B

Justification:

$$\left(2^{(2^x)}\right)^c = 2^{(2^{x+1})}$$

$$2^{c(2^x)} = 2^{(2^{x+1})}$$ since $\left(a^x\right)^y = a^{xy}$

$$c(2^x) = 2^{x+1}$$ since $a^x = a^y$ if and only if $x = y$

$$c = 2^{x+1-x}$$ divide both sides by 2^x (then subtract exponents)

$$c = 2$$

Exponent Laws XIV

Simplify the following: $$\frac{100^2 - 99^2}{(100+99)^2}$$

A. 199^2
B. 199
C. 1
D. 199^{-1}
E. 199^{-2}

Difference of squares:

$$a^2 - b^2 = (a+b)(a-b)$$

Press for hint

Solution

Answer: D

Justification: The numerator is a difference of squares:

$$a^2 - b^2 = (a+b)(a-b)$$

$$\frac{(100+99)(100-99)}{(100+99)^2} = \frac{(100-99)}{(100+99)^{2-1}} = \frac{1}{199} = 199^{-1}$$

OR

$$\frac{(100+99)(100-99)}{(100+99)(100+99)} = \frac{(100-99)}{(100+99)} = \frac{1}{199} = 199^{-1}$$