a place of mind

FACULTY OF EDUCATION

Department of
Curriculum and Pedagogy

Mathematics
Probability: Combinations

Science and Mathematics
Education Research Group

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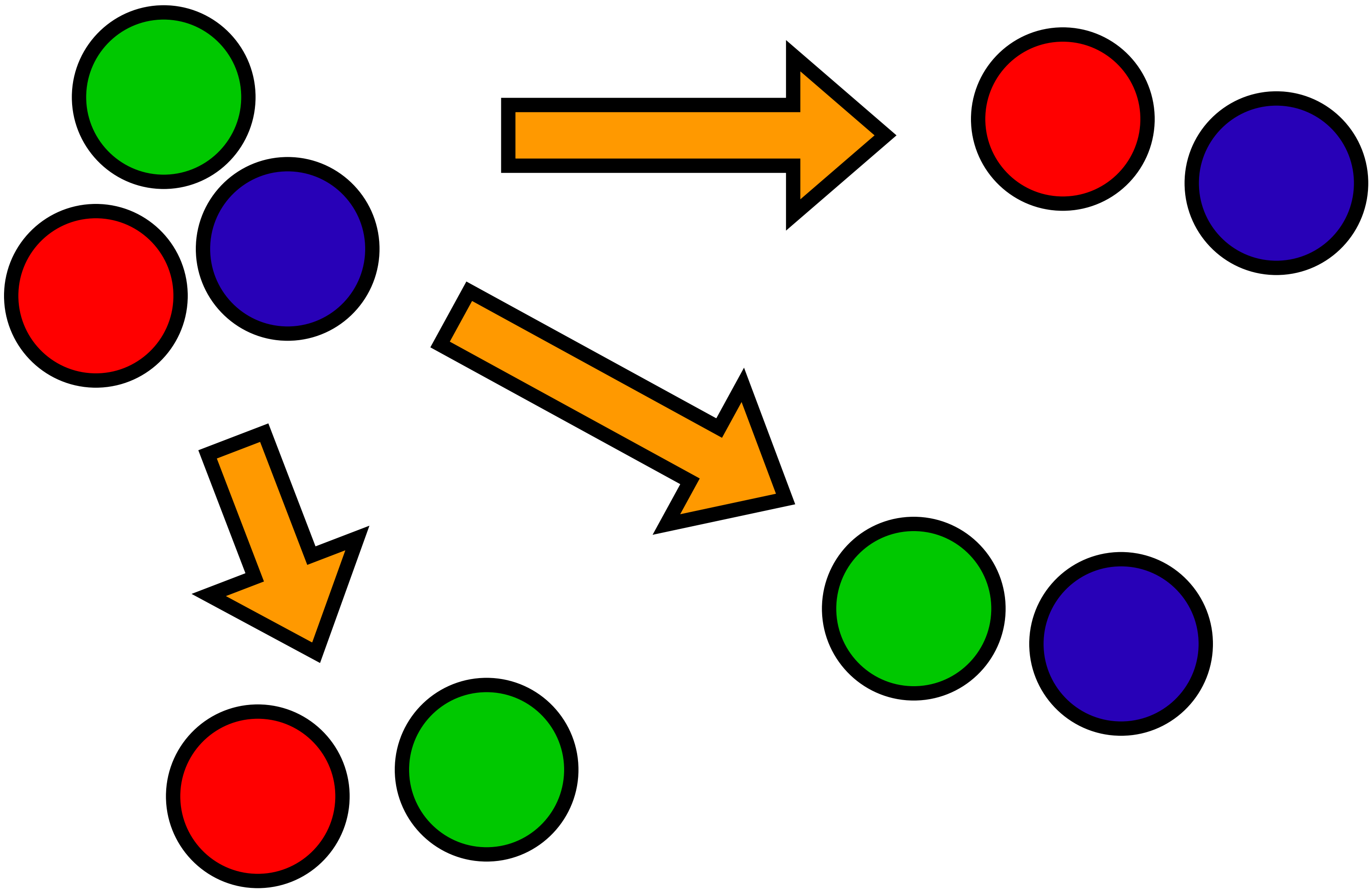
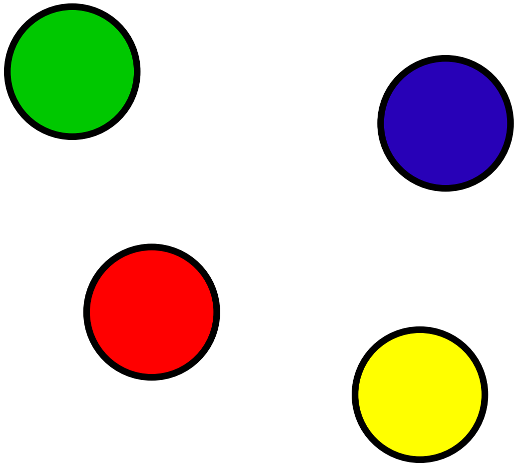
Combinations



Combinations I

Consider a bag of 4 candies, each a different flavour. Bob reaches into the bag and picks out 3 candies at the same time. How many different ways could Bob have chosen his candies?

A. 1
B. 4
C. 8
D. 12
E. 24



Solution

Answer: B

Justification: Let the four different flavours be A, B, C, and D. If Bob picks 3 out of the 4 flavours, there will be 1 flavour remaining in the bag. Since there are only 4 flavours, there are only 4 different ways to pick the left out candy.

The possible combinations are:

ABC, ABD, ACD, BCD

In this case, only the members of the final set of candies matter; the order in which the candies are drawn does not. All permutations of ABC (ABC, ACB, BAC, BCA, CAB, CBA) are considered the same.

Combinations II

Consider a bag of 100 candies. There are 98 blue candies, 1 red candy, and 1 yellow candy. Bob picks 18 candies from the bag. How many different combinations of candies can Bob finish with?

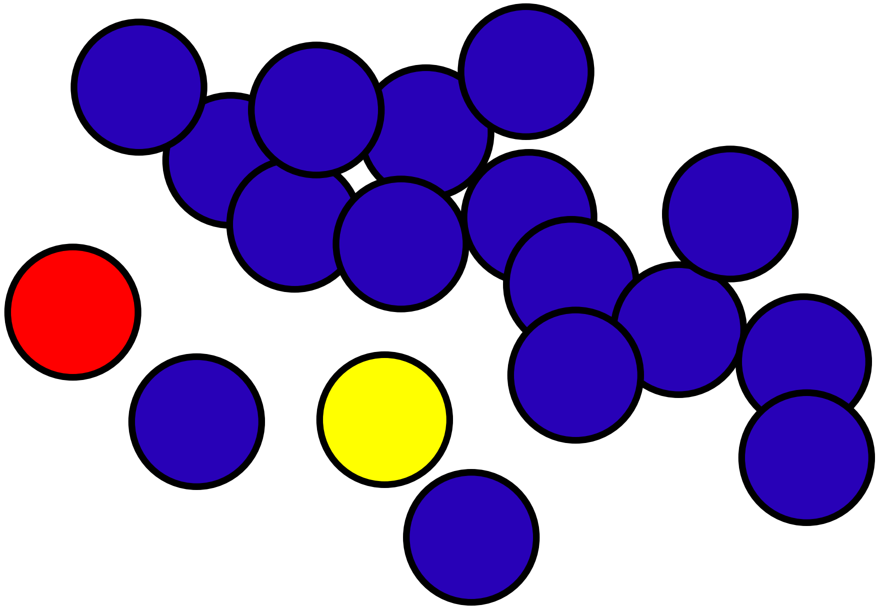
A. 4

B. $\frac{98!}{84!}$

C. $\frac{100!}{84!}$

D. ${}_{100}P_{18}$

E. ${}_{100}P_{98}$



Solution

Answer: A

Justification: Even though there are many blue candies, every time Bob chooses 18 blue candies counts as the same combination. Therefore the only combinations are:

18 blue candies

17 blue candies, 1 red candy

17 blue candies ,1 yellow candy

16 blue candies, 1 red candy, 1 yellow candy

Combinations III

Consider a selection of *n* different objects and we choose *r* of them. If the order in which we select the objects does not matter, how many different ways can the selection be made?

A. $\frac{n!}{r!}$

B. $\frac{n!}{(n-r)!}$

C. $\frac{{}_nP_r}{n!}$

D. $\frac{{}_nP_r}{r!}$

E. $\frac{r!n!}{(n-r)!}$

Press for hint

Hint: ${}_nP_r$ is the number of ways the objects can be selected when order matters. Divide this answer by the number of ways a set of objects can be rearranged into a different order.

Solution

Answer: D

Justification: Let the number of combinations be denoted as ${}_nC_r$. A single selection can be rearranged r! different ways in order to give all the ways the same selection can be made, but in a different order. So if we multiply ${}_nC_r$ by the number of ways we can rearrange r slots, this will equal the number of ways we can permute n objects in r slots. Therefore,

$$\left({}_nC_r\right)\cdot r! = {}_nP_r$$

$${}_nC_r = \frac{{}_nP_r}{r!} = \frac{n!}{(n-r)!\,r!}$$

Another common notation for ${}_nC_r$ is: ${}_nC_r = \binom{n}{r}$

Combinations IV

How many ways can 4 candies be chosen from a bag containing 6 different candies?

$$_nC_r = \frac{_nP_r}{r!} = \frac{n!}{(n-r)!r!}$$

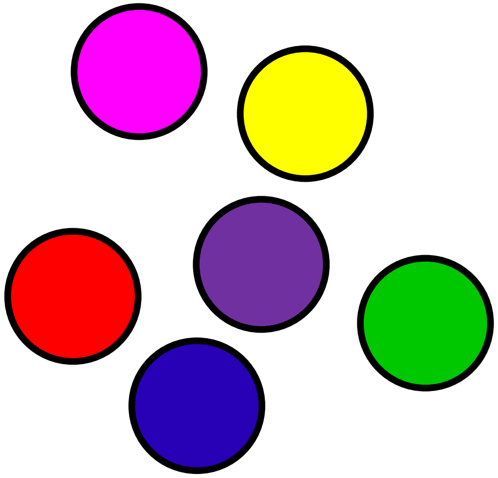
A. 5

B. 15

C. 24

D. 90

E. 360



Solution

Answer: B

Justification: If we first assume that the order in which the candies are selected matters, there are $_6P_4$ or 360 different permutations.

If we divide this answer by the number of ways we can rearrange 4 slots, this will remove all the duplicated answers. The 4 slots can be rearranged 4! = 24 ways, so there are only 360/24 = 15 different combinations.

$$_6C_4 = \frac{_6P_4}{4!} = \frac{6!}{2!4!} = \frac{6(5)}{2} = 15$$

Combinations V

There are 20 people in a boardroom. Every person must handshake with every other person in the room. How many handshakes are done in total?

A. 40
B. 190
C. 380
D. 400
E. 20!

Solution

Answer: B

Justification: Label the 20 people from A to T. Let a selection of 2 letters denote that the two people handshake (for example, AB = person A and person B handshake). The number of ways we can select 2 letters from a group of 20 unique letters is the number of ways we can choose 2 people to handshake. Since order does not matter and every person is unique, the number of handshakes done in total (the number of combinations of 2 letters) is:

$$_{20}C_2 = \frac{20!}{(20-2)!2!} = \frac{20(19)(18)!}{18!2!} = \frac{20(19)}{2} = 190$$

Combinations VI

There are 8 multiple choice clicker questions, each with 5 possible answers. Sam wants to know how many different ways the questions can be answered on the test. His answer is shown below. Is he correct?

There are a total of 40 possible answers to the clicker questions. Out of these answers, 8 must be chosen as the answers to the 8 multiple choice questions. Therefore the number of ways is:

$$\binom{40}{8} = 76904685$$

A. Yes

B. No, the answer is too large

C. No, the answer is too small

Solution

Answer: B

Justification: There are 5 ways to choose the answer to the first question, 5 ways to choose the answer the second question, and so on. Multiplying these answers together give:

$5^8 = 390625$

Notice that these selections are done independently of each other. Sam's answer is the number of ways we can choose 8 answers from a single question with 40 possible answers.

Combinations VII

Jeremy and Kevin are arguing over how many ways 6 winning lottery numbers can be drawn from a group of 48. Who is correct?

Jeremy*: Every choice is unique and the order that the numbers are drawn does not matter, so the number of ways 6 winning numbers can be chosen is* ${}_{48}C_{6}$.

Kevin*: Every time 6 numbers are drawn, there are always 42 losing numbers (the numbers that were not drawn). These 42 numbers can be chosen* ${}_{48}C_{42}$ *ways.*

A. Jeremy is correct

B. Kevin is correct

C. Both are correct

D. Neither are correct

Solution

Answer: C

Justification: Both $_{48}C_6$ and $_{48}C_{42}$ give the same answer: 12271512 different ways to choose the lottery numbers.

Whenever *r* objects are chosen from a selection of *n* objects, there are (n-r) objects that are not chosen. Instead of determining how many ways we can choose the selected numbers, it is the same to choose the objects that were not selected.

Additional activities: Prove that $_nC_r = {}_nC_{n-r}$ using the definition of factorials.

Combinations VIII

What value of *r* will give the maximum value for $_{100}C_r$?

(What group size will give the most combinations if we need to choose a group out of 100 people?)

A. r = 1

B. r = 100

C. r = 50 or r = 51 (both these values give the maximum)

D. r = 49 or r = 50 (both these values give the maximum)

E. r = 50

Solution

Answer: E

Justification: When r = 1, ${}_{100}C_r$ = 100 since there are only 100 ways to pick 1 person from 100 people.

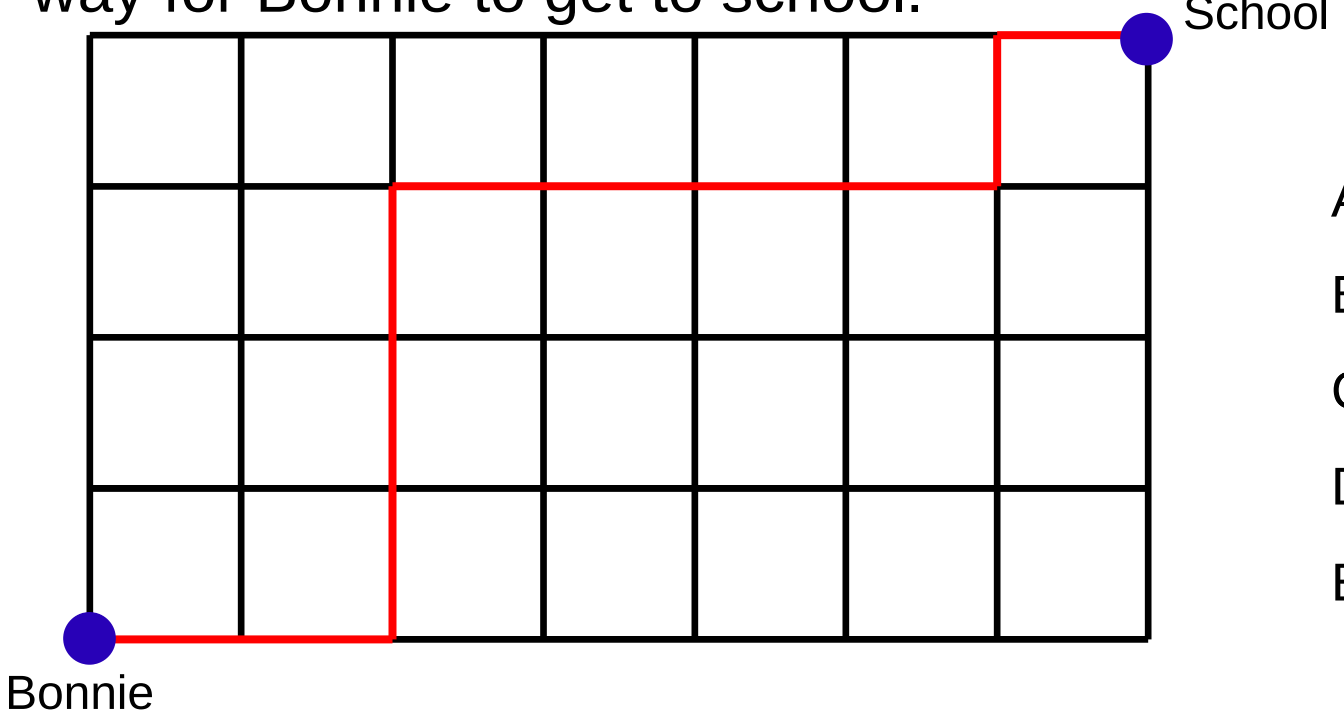
When r = 100, ${}_{100}C_r$ = 1 since there is only 1 way to pick 100 people from 100 people.

${}_{100}C_{50}$ is not the same as ${}_{100}C_{51}$. In addition, ${}_{100}C_{50}$ is not the same as ${}_{100}C_{49}$. From the previous question, ${}_nC_r = {}_nC_{n-r}$ but when r = 50, n-r = 100-50 = 50. Since ${}_{100}C_{50}$ is not equal to any other value, answers C and D are false.

The middle number between 0 to 100 gives the maximum value, ${}_{100}C_{50}$.

Combinations IX

The blocks on a street are shown below. How many ways can Bonnie get from her house to school? She should only travel 11 blocks. The red path shows one possible way for Bonnie to get to school.



A. 330

B. 990

C. 1980

D. 7920

E. 1663200

Solution

Answer: A

Justification: In every single path to school, Bonnie must travel 7 blocks east and 4 blocks north. A path can be denoted by a string of E's and N's: EEEEEENNNNE (first travel 6 blocks east, then 4 blocks north, and finally 1 block east).

There are 11 total positions for the letters E and N, respectively. The number of ways we can choose a position for the E's is ${}_{11}C_7$. The positions that were not chosen will be filled with N's. This gives every unique path from Bonnie's house to her school.

$$ {}_{11}C_7 = \frac{11!}{(11-7)!7!} = \frac{11(10)(9)(8)(7!)}{4!7!} = \frac{11(10)(9)(8)}{24} = 30(11) = 330 $$

Alternative Solution

Answer: A

Justification: In every single path to school, Bonnie must travel 7 blocks east and 4 blocks north. A path can be denoted by a string of E's and N's: EEEEEENNNNE (first travel 6 blocks east, then 4 blocks north, and finally 1 block east).

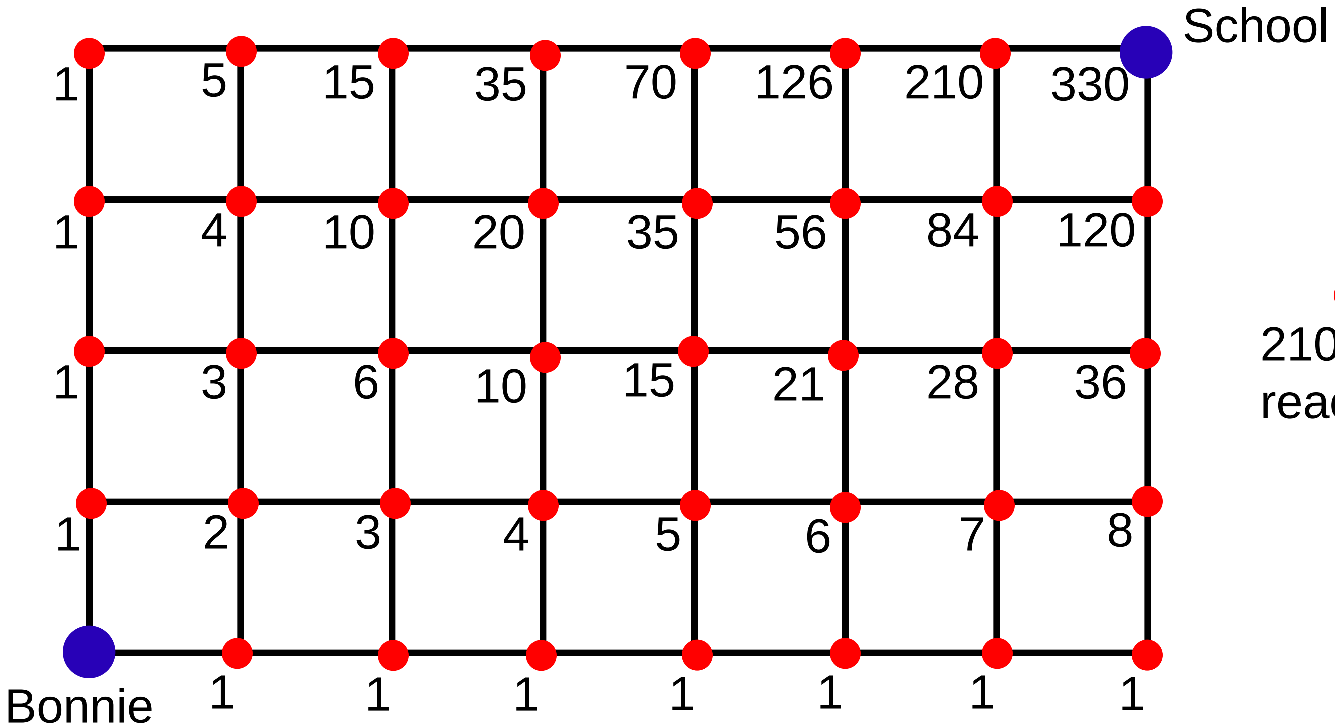
Using permutations with repetitions: There are 11 letters, with 7 E's repeated and 4 N's repeated. The number of ways the repeated letters can be permuted must be divided:

$$\frac{11!}{4!7!} = \frac{11(10)(9)(8)}{24} = 30(11) = 330$$

Alternative Solution II

Answer: A

Justification: The question can also be solved by considering how many different ways each intersection can be reached by adding the ways the 2 streets leading to it can be reached.



Add the ways leading to this point can be reached
210 + 120 = 330