a place of mind

FACULTY OF EDUCATION

Department of
Curriculum and Pedagogy

# Mathematics
# Conditional Probability

Science and Mathematics
Education Research Group

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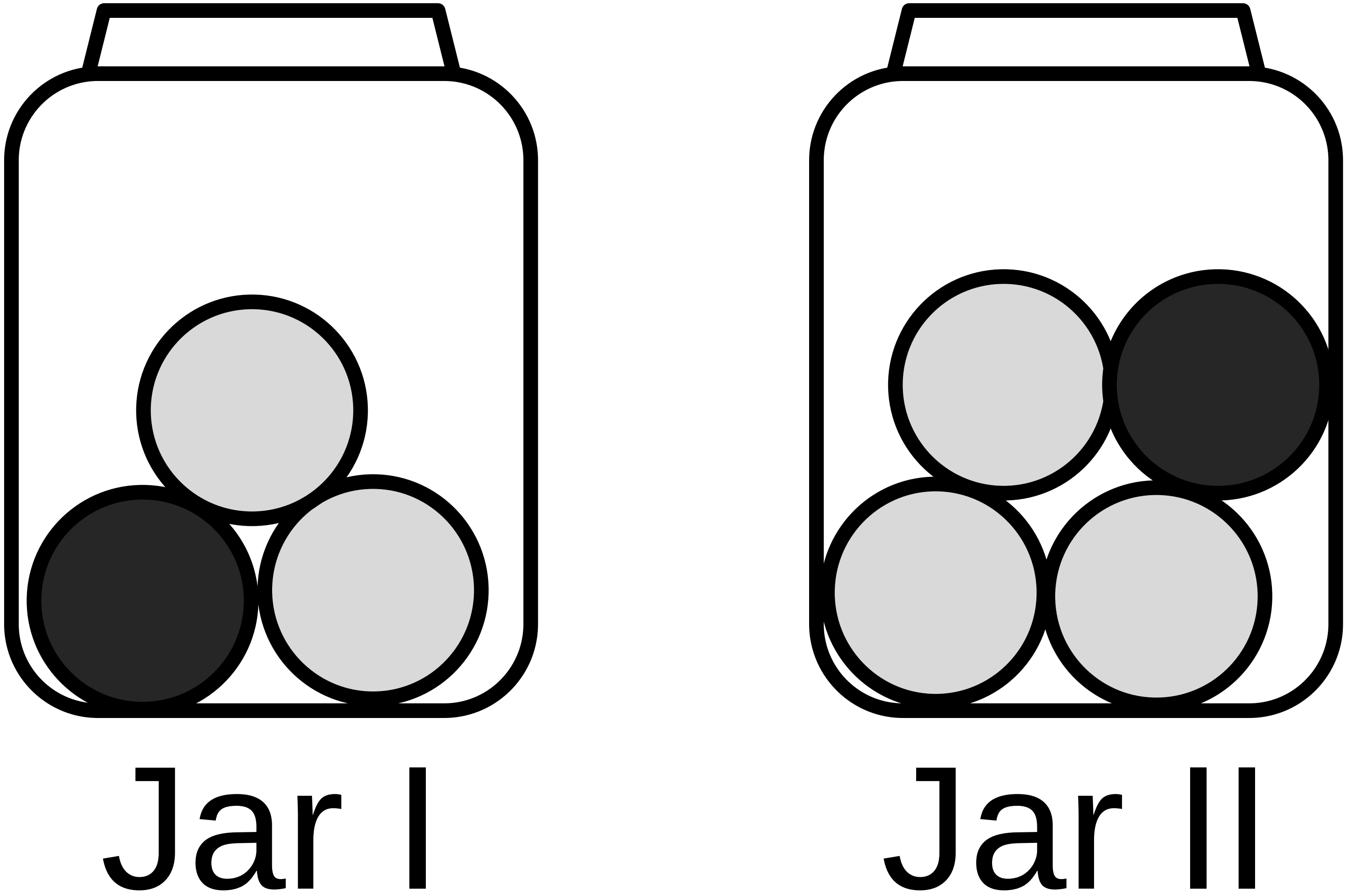
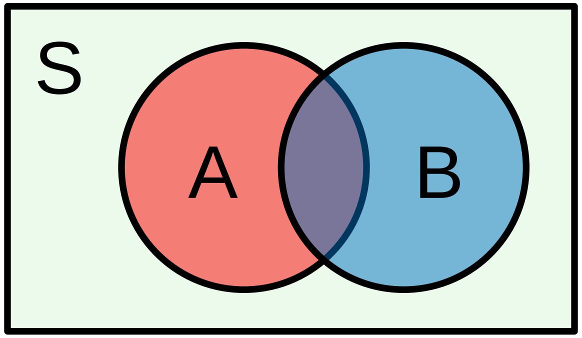
# Conditional Probability



# Conditional Probability I

Consider the following sample space and two events depicted in the diagram. What is the probability of A, if we know event B has occurred?




A. $P(\text{A})$

B. $P(\text{A and B})$

C. $P(\text{A}) + P(\text{A and B})$

D. $P(\text{A}) \cdot P(\text{A and B})$

E. $\dfrac{P(\text{A and B})}{P(\text{B})}$

*Press for hint*

Imagine that B is the sample space. What fraction of B is covered by A?

# Solution

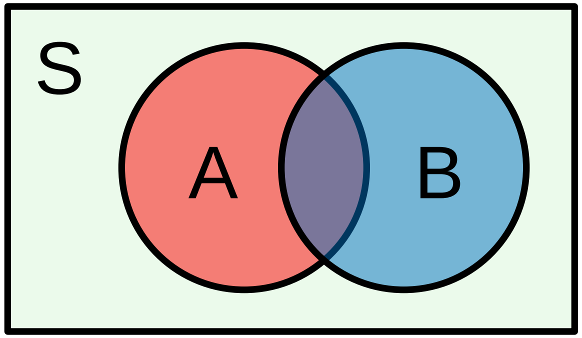
**Answer:** E

**Justification:** Conditional probability is the probability that A occurs, given B has occurred. It is denoted $P(\mathrm{A}\,|\,\mathrm{B})$ – *"the probability of A given B"*.

If we assume B has occurred, our sample space is limited to the events of B. Therefore, if A also occurs, it must be that both A and B occurred.

This is the fraction of B covered by A.

$$P(\mathrm{A}\,|\,\mathrm{B}) = \frac{P(\mathrm{A\ and\ B})}{P(\mathrm{B})}$$




Notice that the probability of A and B is the probability of B times the probability of A given B.

$$P(\mathrm{A\ and\ B}) = P(\mathrm{B}) \cdot P(\mathrm{A}\,|\,\mathrm{B})$$

# Conditional Probability II

Seven coins are flipped. What is the probability that all seven coins land on heads, given that the first six coins landed on heads?

$$P(\text{All heads} \mid \text{First 6 are heads}) =$$

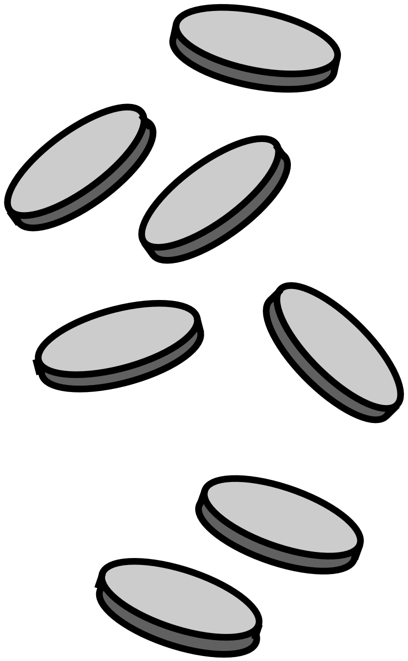
A. $\frac{1}{2}$

B. $\frac{1}{2^7}$

C. $1-\frac{2^6}{2^7}$

D. $1-\frac{1}{2^7}$

E. None of the above



# Solution

**Answer:** A

**Justification:** If it is given that the first 6 coins land on heads, then only the seventh coin determines if all the coins are heads. The probability that the last coin is heads is 50%. Since the outcome of this coin flip determines whether all seven are heads, then the probability that all 7 are heads, given the first 6 are heads, is also 50%.

*Answer continues on the next slide*

# Solution

If this question was done using the formulas:

A = All 7 coins are heads

B = First 6 coins are heads

The sample space has $2^7$ elements since we are flipping 7 coins. The set of outcomes we need to find are:

A and B = { HHHHHHH }

B = { HHHHHHT, HHHHHHH }

$$P(\mathrm{A} \mid \mathrm{B}) = \frac{P(\mathrm{A \text{ and } B})}{P(B)} = \frac{\frac{1}{2^7}}{\frac{2}{2^7}} = \frac{1}{2}$$

Note: Since A is a subset of B, *P*(A and B) = *P*(A) .

# Conditional Probability III

Two coins are flipped. What is the probability that 2 heads are landed, given that at least 1 of the coins landed on heads?

$P(2 \text{ heads} \mid \text{At least 1 head}) =$

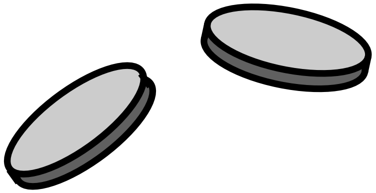
A. $\frac{1}{2}$

B. $\frac{1}{3}$

C. $\frac{1}{4}$

D. $\frac{3}{4}$

E. $\frac{2}{3}$



# Solution

**Answer:** B

**Justification:** Let A = 2 heads are landed, B = At least 1 head is landed.

Since A is a subset of B (2 heads implies at least 1 head is landed), the probability of A and B the same as P(A).

$$P(\text{A and B}) = P(\text{A}) = \frac{1}{4}$$

P(B) can be found in several ways. Since the sample space is so small, it is easiest just to count all the outcomes:

$$\text{B} = \{\text{HH}, \text{HT}, \text{TH}\} \Rightarrow P(\text{B}) = \frac{3}{4}$$

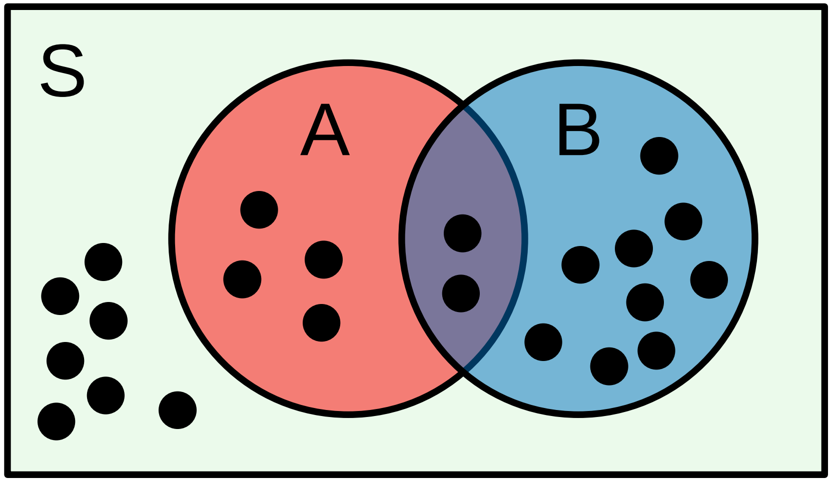
Using the definition for conditional probability:

$$P(\text{A} \mid \text{B}) = \frac{P(\text{A and B})}{P(B)} = \frac{1}{3}$$

# Conditional Probability IV

Consider the following sample space. Each dot represents an equally likely outcome.

Determine $P(\mathrm{B} \mid \mathrm{A})$.




A. $\frac{1}{3}$

B. $\frac{2}{3}$

C. $\frac{2}{9}$

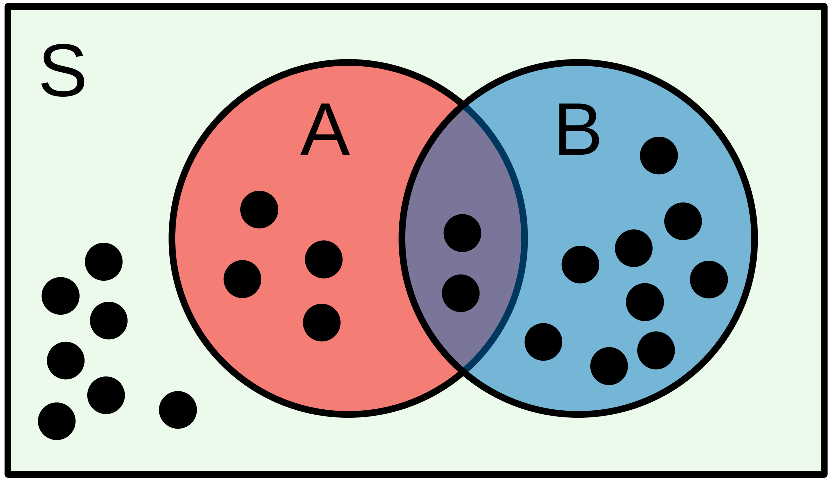
D. $\frac{7}{9}$

E. $\frac{1}{2}$

# Solution

**Answer:** A

**Justification:** We must find the probability a dot is in B, given that the dot is in A.




This question can be solved by only looking at the dots in the red circle, since it is given that A has occurred.

Find the fraction of the dots in A that are also in B:

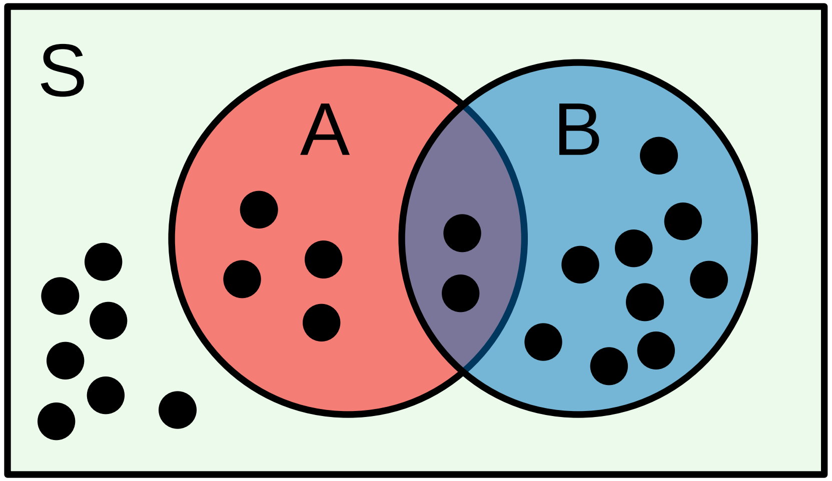
$$P(\mathrm{B}\,|\,\mathrm{A}) = \frac{\text{Dots in A and B}}{\text{Total dots in A}} = \frac{2}{6} = \frac{1}{3}$$

*Answer continues on the next slide*

# Solution

**Answer:** A

**Justification:**




If this question were solved by the formula for conditional probability, we would have to count all the dots in the sample space:

$$P(\text{B and A}) = \frac{2}{22}$$

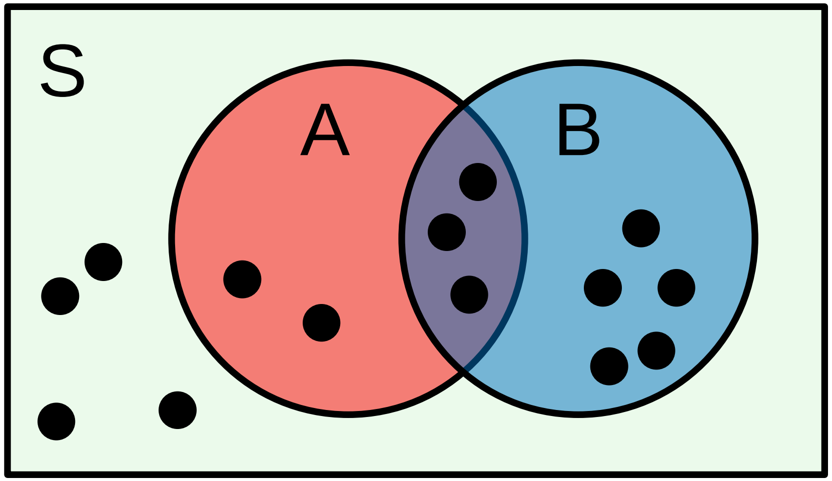
$$P(\text{A}) = \frac{6}{22}$$

$$P(\text{B} \mid \text{A}) = \frac{P(\text{B and A})}{P(\text{A})} = \frac{\frac{2}{22}}{\frac{6}{22}} = \frac{1}{3}$$

# Conditional Probability V

Consider the following sample space. Each dot represents an equally likely outcome.

Determine $P(\overline{\mathrm{B}} \mid \overline{\mathrm{A}})$.




A. $\frac{4}{5}$

B. $\frac{2}{3}$

C. $\frac{1}{3}$

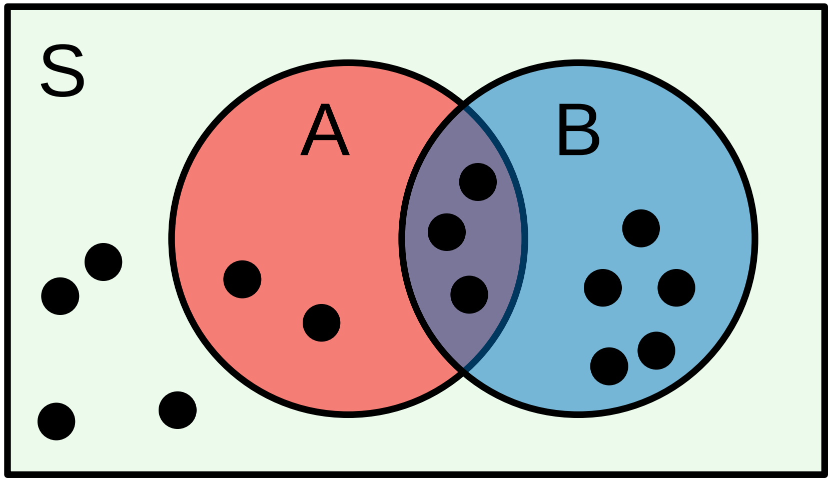
D. $\frac{4}{9}$

E. $\frac{5}{9}$

# Solution

**Answer:** D

**Justification:** We must find the probability that a dot is not in B, given that the dot is also not in A.




Consider only the 9 dots not in A. Of these, only 4 are not in B. Therefore:

$$P(\overline{\mathrm{B}} \mid \overline{\mathrm{A}}) = \frac{P(\overline{\mathrm{B}} \text{ and } \overline{\mathrm{A}})}{P(\overline{\mathrm{A}})} = \frac{4}{9}$$

Note:

$$\left( P(\overline{\mathrm{B}} \text{ and } \overline{\mathrm{A}}) = \frac{4}{14}, \ P(\overline{\mathrm{A}}) = \frac{9}{14} \right)$$

*Notice that* $P(\mathrm{B} \mid \mathrm{A}) = \frac{3}{5}$ *and* $P(\overline{\mathrm{B} \mid \mathrm{A}}) = 1 - P(\mathrm{B} \mid \mathrm{A}) = \frac{2}{5}$.

*This shows that* $P(\overline{\mathrm{B}} \mid \overline{\mathrm{A}}) \neq P(\overline{\mathrm{B} \mid \mathrm{A}})$.

# Conditional Probability VI

Consider the following two definitions of conditional probability:

1) $P(\mathrm{A} \mid \mathrm{B}) = \dfrac{P(\mathrm{A\ and\ B})}{P(\mathrm{B})}$ 2) $P(\mathrm{B} \mid \mathrm{A}) = \dfrac{P(\mathrm{B\ and\ A})}{P(\mathrm{A})}$

Which one of the following formulas correctly relate $P$(A | B) with $P$(B | A)?

A. $P(\mathrm{A} \mid \mathrm{B}) = P(\mathrm{B} \mid \mathrm{A})$

B. $P(\mathrm{A} \mid \mathrm{B}) = \dfrac{P(\mathrm{A})}{P(\mathrm{B})} P(\mathrm{B} \mid \mathrm{A})$

C. $P(\mathrm{A} \mid \mathrm{B}) = \dfrac{P(\mathrm{B})}{P(\mathrm{A})} P(\mathrm{B} \mid \mathrm{A})$

D. $P(\mathrm{A} \mid \mathrm{B}) = \dfrac{P(\mathrm{A\ and\ B})}{P(\mathrm{B\ and\ A})} P(\mathrm{B} \mid \mathrm{A})$

E. $P(\mathrm{A} \mid \mathrm{B})$ and $P(\mathrm{B} \mid \mathrm{A})$ cannot be related.

# Solution

**Answer:** B

**Justification:** The only term common in the two formulas is P(A and B).

$$1) \quad P(\mathrm{A} \mid \mathrm{B}) = \frac{P(\mathrm{A\ and\ B})}{P(\mathrm{B})} \qquad 2) \quad P(\mathrm{B} \mid \mathrm{A}) = \frac{P(\mathrm{B\ and\ A})}{P(\mathrm{A})}$$

Rewriting formula 2 to isolate P(A and B):

$$2) \quad P(\mathrm{A\ and\ B}) = P(\mathrm{B} \mid \mathrm{A})P(\mathrm{A})$$

Substituting 2 into 1:

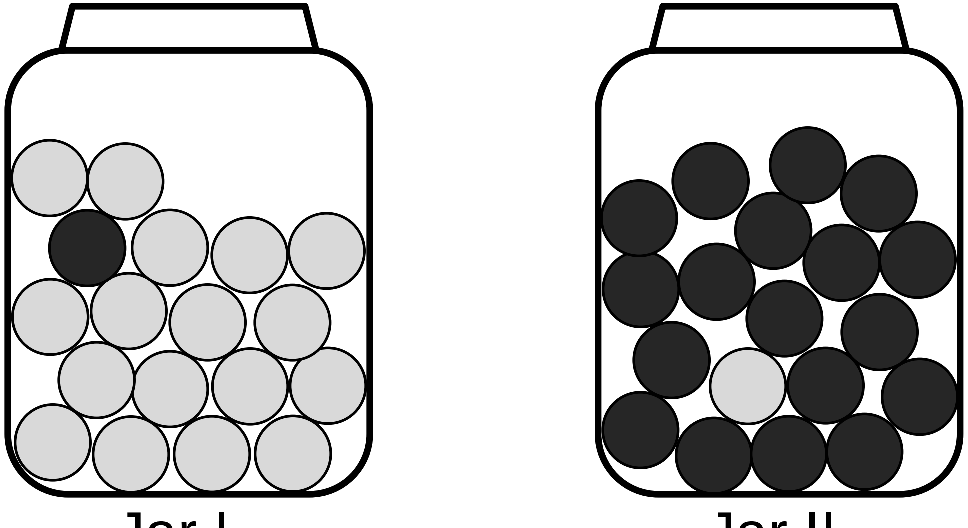
$$\boxed{P(\mathrm{A} \mid \mathrm{B}) = \frac{P(\mathrm{A})}{P(\mathrm{B})} P(\mathrm{B} \mid \mathrm{A})}$$

This is known as *Baye's Theorem*, which relates P(A|B) with P(B|A).

# Conditional Probability VII

Consider 2 jars each filled with 100 marbles. Jar I has 99 white marbles and 1 black marble, while Jar II has 99 black marbles and 1 white marble. A jar is first selected at random, and then a marble is pulled out.

What is the probability the marble is white?




A. $\frac{49}{100}$

B. $\frac{1}{2}$

C. $\frac{51}{100}$

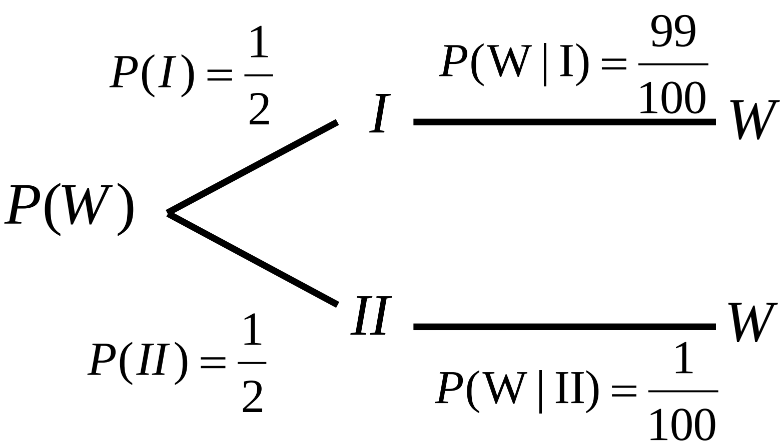
D. $\frac{99}{101}$

E. $\frac{1}{101}$

# Solution

**Answer:** B

**Justification:** There are two ways a white marble can be drawn, either through Jar I or Jar II (see tree diagram). The probability of white is therefore the sum of ‘the probability of white and Jar I’ and ‘the probability of white and Jar II’. We use conditional probability to find these two separate probabilities:



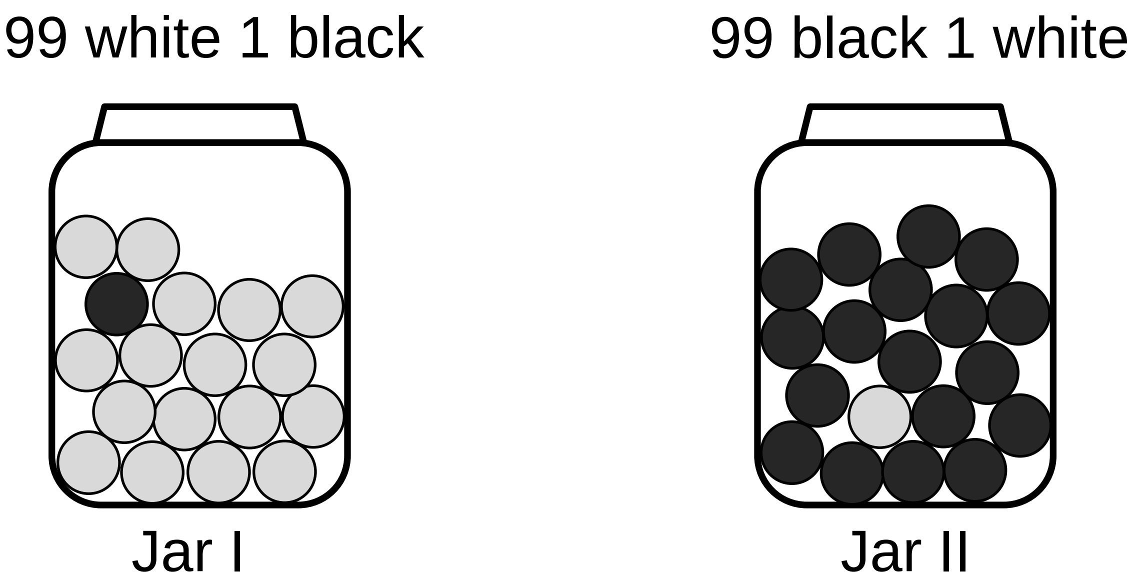

$$\begin{aligned} P(\mathrm{W}) &= P(\mathrm{W\ and\ I}) + P(\mathrm{W\ and\ II}) \\ &= P(\mathrm{I})P(\mathrm{W} \mid \mathrm{I}) + P(\mathrm{II})P(\mathrm{W} \mid \mathrm{II}) \\ &= \frac{1}{2} \cdot \frac{99}{100} + \frac{1}{2} \cdot \frac{1}{100} \\ &= \frac{1}{2} \end{aligned}$$

*See next slide for an alternative solution*

# Solution

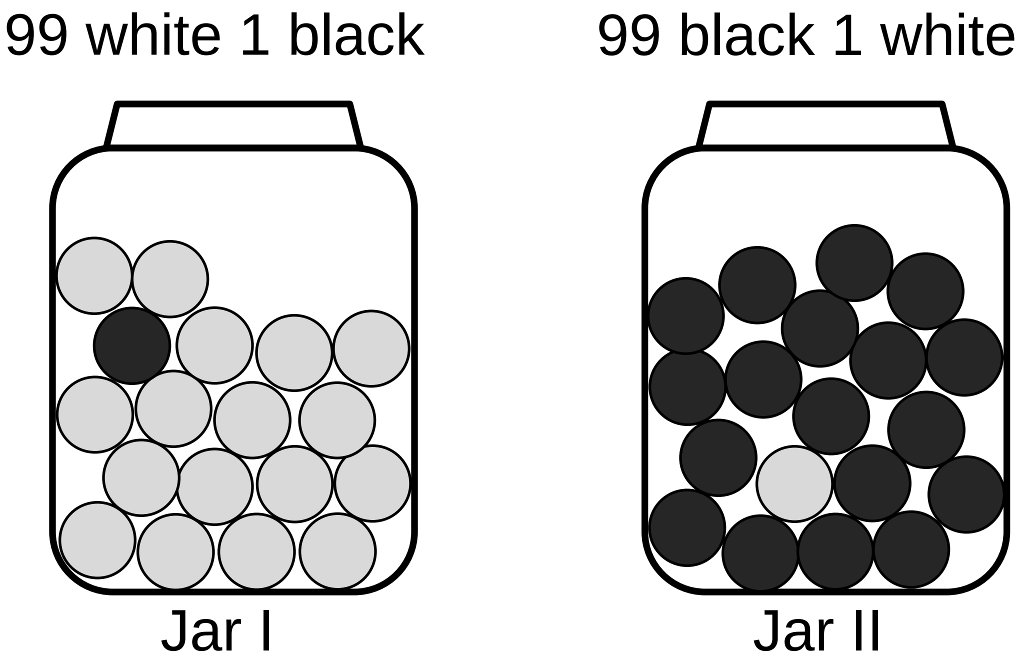
**Answer:** B

**Justification:** It is also possible to solve this question without any calculations by realizing that there would be no difference if the question asked for the probability of selecting a black marble. Since the conditions for selecting black and white marbles are the same, you can expect that the probability of selecting either is 50%.



# Conditional Probability VIII

If the marble selected is white, what is the probability that it came from jar II?




A. $\frac{1}{2}$

B. $\frac{1}{99}$

C. $\frac{1}{100}$

D. $\frac{1}{101}$

E. $\frac{2}{101}$

# Solution

**Answer:** C

**Justification:** The key to solving this problem is recognizing that we are looking to calculate the probability that Jar II was selected, given that the selected marble was white.

Using Baye's Theorem:

$$P(\mathrm{II} \mid \mathrm{W}) = \frac{P(\mathrm{II})P(\mathrm{W} \mid \mathrm{II})}{P(\mathrm{W})} = \frac{\frac{1}{2} \cdot \frac{1}{100}}{\frac{1}{2}} = \frac{1}{100}$$

Note: P(W) was found from the previous question

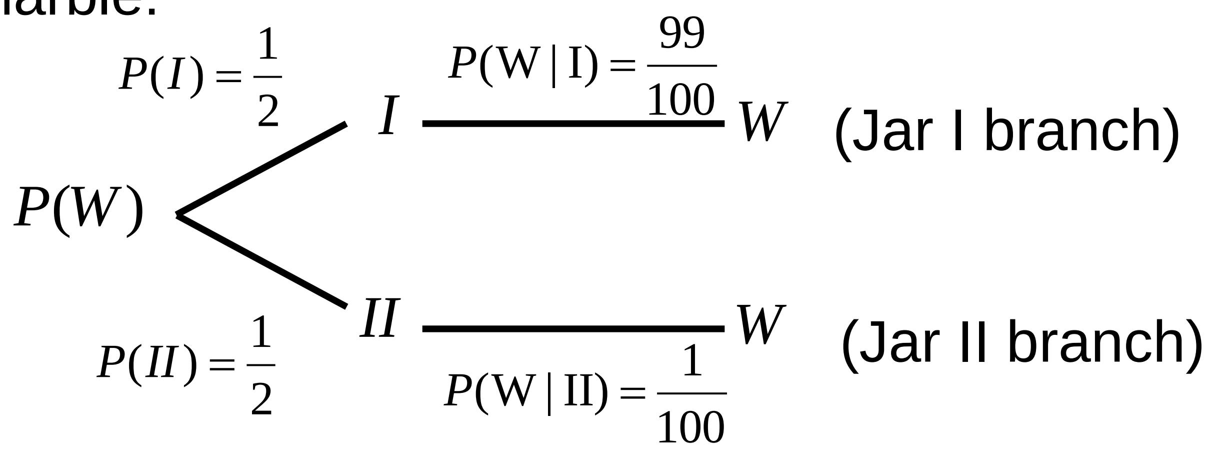
This means that 99% of the time a white ball comes from Jar I, although 1% of the time a white ball will come from Jar II.

*Explanation continues onto the next slide*

# Solution

**Answer:** C

**Justification:** Baye's theorem is easier understood by considering the tree diagram. Consider all the branches that can lead to a white marble:

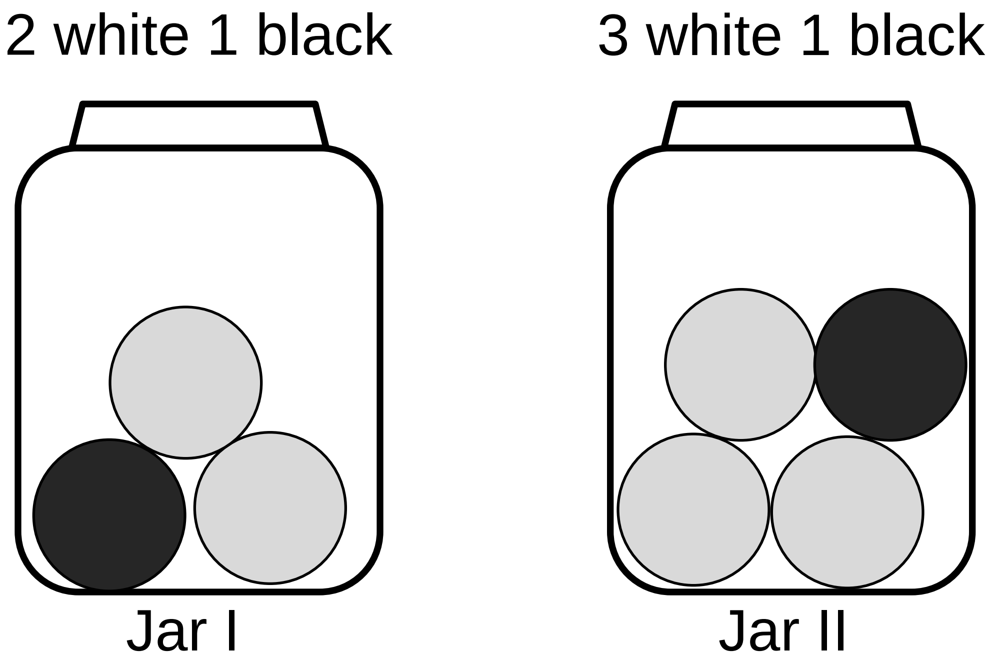



The probability it came from Jar II is the Jar II branch divided by the sum of all the branches.

$$P(\text{II} \mid \text{W}) = \frac{\text{Jar II branch}}{\text{All branches}} = \frac{P(\text{II})P(\text{W} \mid \text{II})}{P(\text{W})} = \frac{1}{100}$$

# Conditional Probability IX

Jar I now has 2 white and 1 black marble while Jar II has 3 white and 1 black marble. After a jar is selected at random, and a marble is pulled out, what is the probability the marble is black?




A. $\frac{1}{4}$

B. $\frac{2}{7}$

C. $\frac{7}{24}$

D. $\frac{1}{3}$

E. $\frac{2}{5}$

# Solution

**Answer:** C

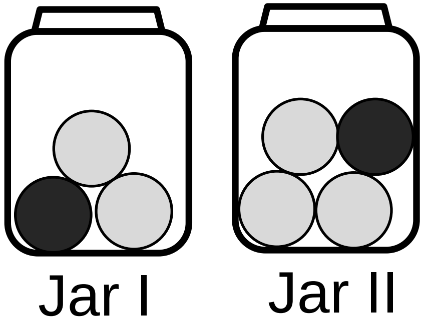
**Justification:** This answer is best solved using a similar calculation done in question 8. The probability that we find a black marble is the sum of the probabilities that the marble is black given each of the jars:

$$\begin{aligned} P(\mathrm{B}) &= P(\mathrm{B\ and\ I}) + P(\mathrm{B\ and\ II}) \\ &= P(\mathrm{I})P(\mathrm{B} \mid \mathrm{I}) + P(\mathrm{II})P(\mathrm{B} \mid \mathrm{II}) \\ &= \frac{1}{2}\cdot\frac{1}{3} + \frac{1}{2}\cdot\frac{1}{4} \\ &= \frac{7}{24} = 29.2\% \end{aligned}$$

Notice that the answer is between 25% (the probability of drawing black from Jar II) and 33% (the probability of drawing black from Jar I). This fact should be used to eliminate answers A, D and E.

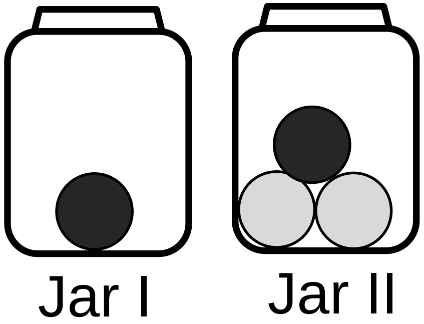
*Answer continues on the next slide*

# Solution Part II




It is a bit harder to see why the answer is not $\frac{2}{7}$, the number of black marbles over the total number of marbles.

Imagine if the marbles were distributed as shown below:

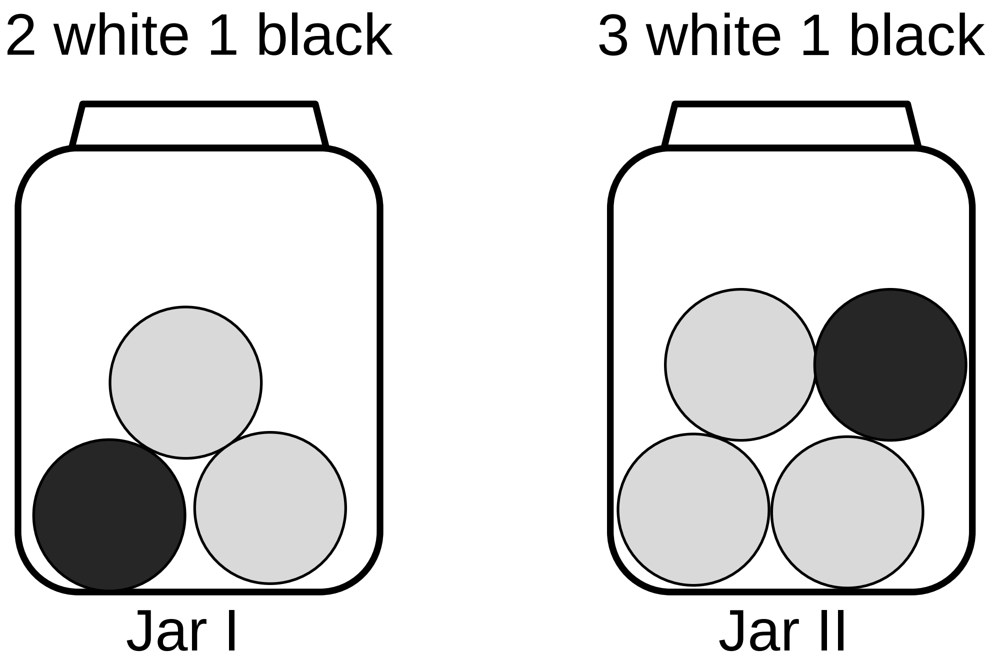



If the above logic is used, the probability of drawing black would be 50%. However, it should be clear that the actual probability is higher than this. At least 50% of the time Jar I is selected, leading to a black marble. If Jar II is selected, there is still a chance the marble is black.

# Conditional Probability X

If the marble selected is black, what is the probability that it came from Jar I?

Review the solution to question 8 if you need help.




A. $\frac{1}{2}$

B. $\frac{1}{3}$

C. $\frac{3}{7}$

D. $\frac{4}{7}$

E. $\frac{5}{7}$

# Solution

**Answer:** D

**Justification:**

Using Baye's Theorem: $P(\mathrm{I}\,|\,\mathrm{B}) = \dfrac{P(\mathrm{I})P(\mathrm{B}\,|\,\mathrm{I})}{P(\mathrm{B})} = \dfrac{\frac{1}{2}\cdot\frac{1}{3}}{\frac{7}{24}} = \dfrac{4}{7}$

Notice that the answer to the previous question was needed to answer this question. Alternatively we may look at the tree diagram. The numerator is the branch with Jar I, while the denominator is the sum of the branches to select black.

$$P(\mathrm{I}\,|\,\mathrm{B}) = \frac{P(\mathrm{I})P(\mathrm{B}\,|\,\mathrm{I})}{P(\mathrm{I})P(\mathrm{B}\,|\,\mathrm{I}) + P(\mathrm{II})P(\mathrm{B}\,|\,\mathrm{II})}$$