a place of mind

FACULTY OF EDUCATION

Department of
Curriculum and Pedagogy

Mathematics
Probability: Events

Science and Mathematics
Education Research Group

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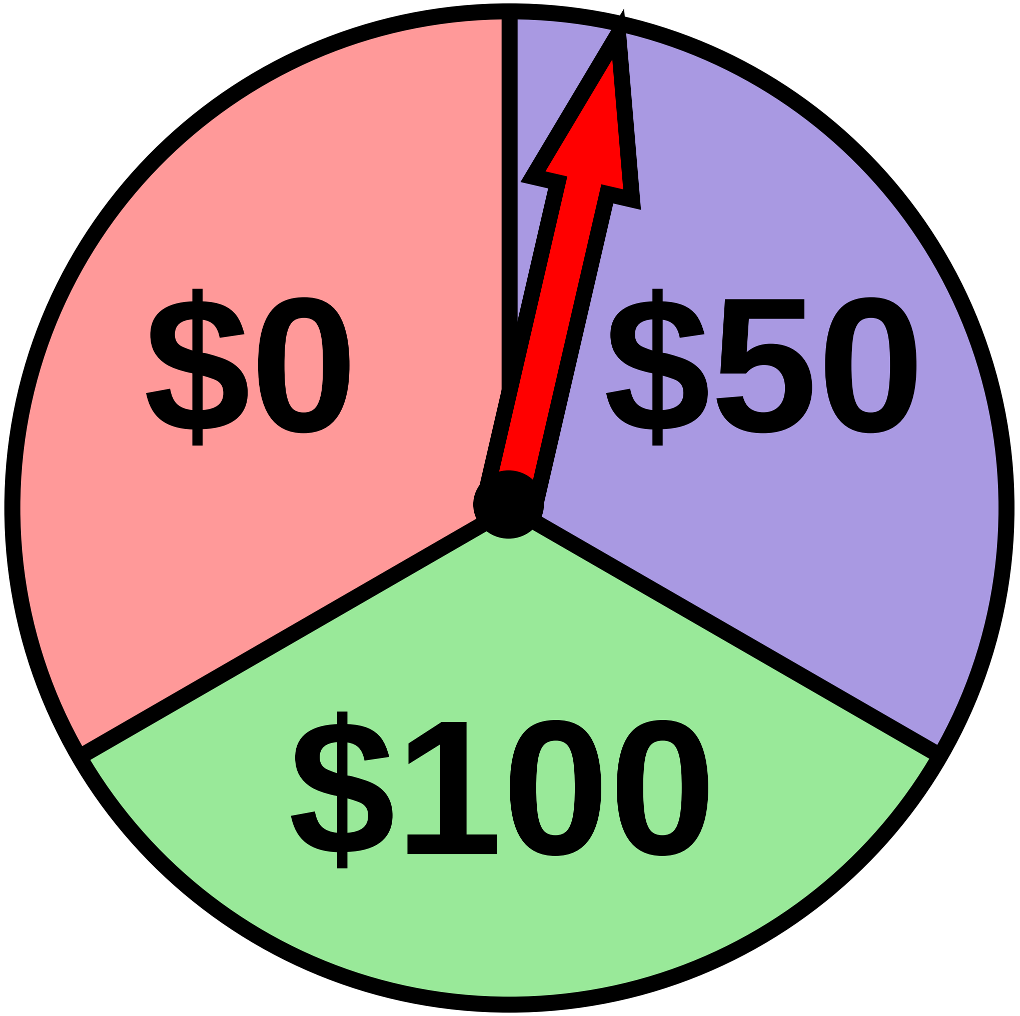
Events

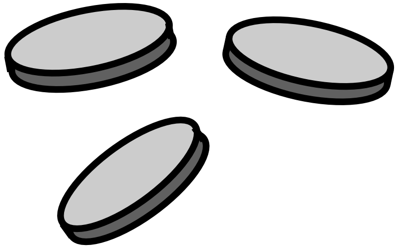


Events I

George tosses a coin three times. After each toss George records that the coin either lands as Heads (H) or Tails (T).

Which of the following sets represents the sample space of all equally likely outcomes?

A. S = {TTT, TTH, THT, HTT, THH, HTH, HHT, HHH}
B. S = {TTT, TTH, THH, HHH}
C. S = {H, T}
D. S = {HT, HT, HT}
E. None of the above



Solution

Answer: A

Justification: There are 2^3 different outcomes after tossing a coin 3 times. The set of all the possible outcomes is:

S = {TTT, TTH, THT, HTT, THH, HTH, HHT, HHH}

Note that S = {TTT, TTH, THH, HHH} may also be used to represent all the possible outcomes if we ignore the order of the results. However, this sample space does not contain all *equally likely* outcomes. There are more outcomes where TTH or THH will be the final result.

Events II

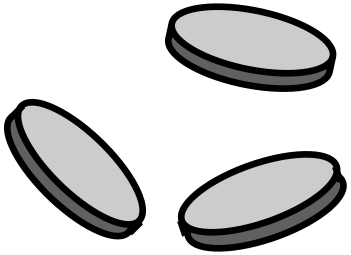
An event is a subset of the sample space of an experiment.

Consider the same experiment as the previous question, where George tosses 3 coins.

Which one of the following statements describes the event:

E = {TTH, THT, HTT}

A. Getting exactly 2 heads
B. Getting exactly 2 tails
C. Getting exactly 1 heads
D. Tossing 3 coins
E. Both B and C



Solution

Answer: E

Justification: Recall that the sample space of the experiment is:

S = {TTT, TTH, THT, HTT, THH, HTH, HHT, HHH}

Event E is a subset of this sample space consisting of:

E = {TTH, THT, HTT}

This event includes all the outcomes where George lands 2 tails and 1 heads. This event can be described by either "Getting exactly 2 tails" or "Getting exactly 1 heads." Getting 2 tails implies that George only landed 1 heads, because the outcome of each coin toss is only 1 of 2 possibilities.

Events III

What is the probability that George lands exactly 2 tails after tossing a coin three times?

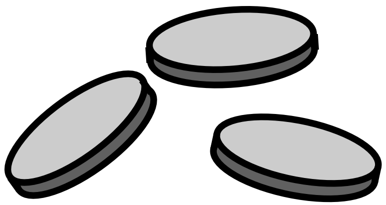
A. P(exactly 2 tails) = 66.6%

B. P(exactly 2 tails) = 50%

C. P(exactly 2 tails) = 37.5%

D. P(exactly 2 tails) = 25%

E. P(exactly 2 tails) = 12.5%



Solution

Answer: C

Justification: There were a total of 8 possible *equally likely* outcomes after flipping a coin 3 times.

S = {TTT, TTH, THT, HTT, THH, HTH, HHT, HHH}

Of these 8 *equally likely* outcomes, there are 3 where George lands exactly two tails.

E = {TTH, THT, HTT}

The probability of landing exactly 2 tails is:

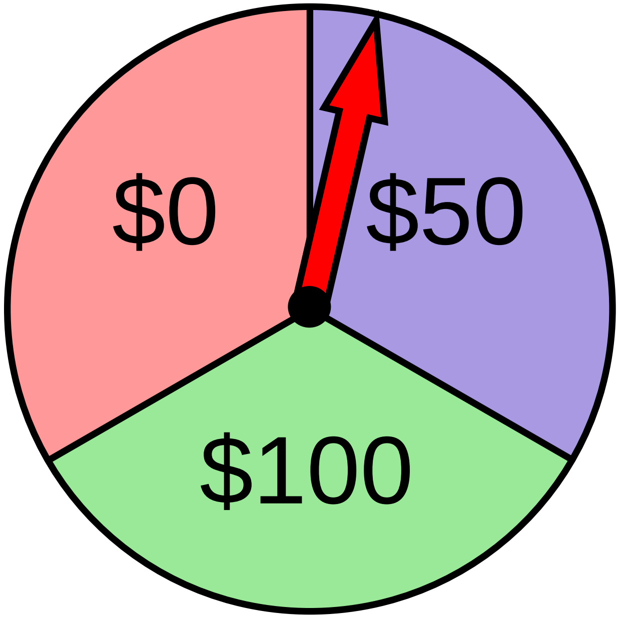
$$P(E) = \frac{\text{number of ways of getting exactly 2 tails}}{\text{total number of outcomes}} = \frac{3}{8} = 37.5\%$$

Events IV

Contestants in a game show spin the wheel shown below twice to determine how much money they win. The sum of both spins is the final amount each contestant wins.

What is the probability that a contestant wins $100?

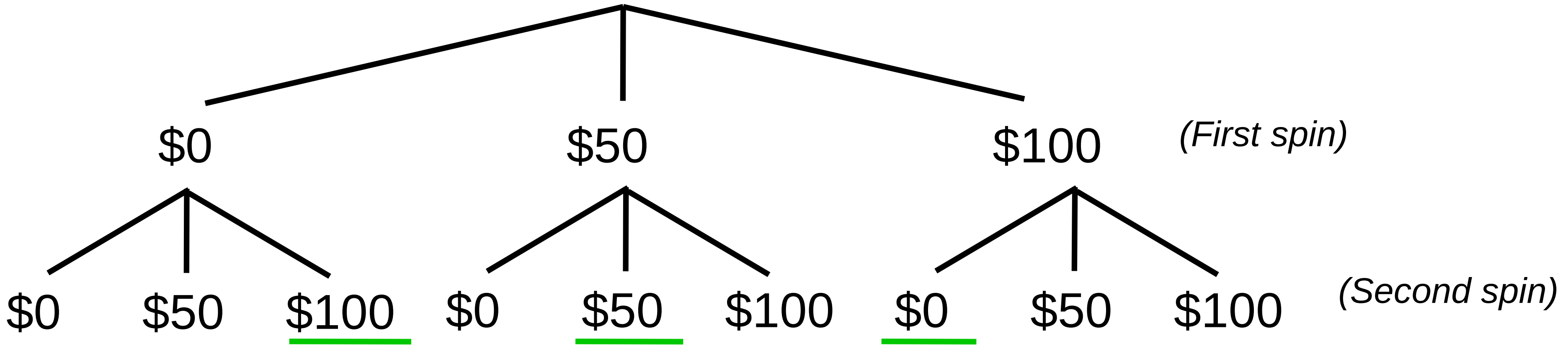
A. P($100) = 75%
B. P($100) = 66.6%
C. P($100) = 50%
D. P($100) = 33.3%
E. P($100) = 11.1%



Solution

Answer: D

Justification: It is helpful to draw a tree diagram to create the sample space of all outcomes:



There are 3 outcomes where the contestants win $100, out of a total of 9 outcomes. The probability to win $100 is therefore:

$$P(\$100) = \frac{\text{ways to win } \$100}{\text{total number of outcomes}} = \frac{3}{9} = 33.3\%$$

Solution II

Answer: D

Justification: The answer can also be solved just by listing out the combinations that only add up to \$100. The only way to finish with \$100 is by the following spin combinations:

First Spin	Second Spin	Total
\$0	\$100	\$100
\$50	\$50	\$100
\$100	\$0	\$100

Since there are 2 spins each with 3 options, the total number of combinations is $3^2 = 9$. The probability of finishing with \$100 is therefore:

$$P(\$100) = \frac{\text{ways to win } \$100}{\text{total number of outcomes}} = \frac{3}{9} = 33.3\%$$

Events V

The winner of a game show spins the wheel once and then picks a ball from the Bonus Box. Two out of nine balls in the Bonus Box contain “x10”, which multiplies the winnings by 10.

What is the probability of $P(\text{Not } \$0 \text{ and x10})$?

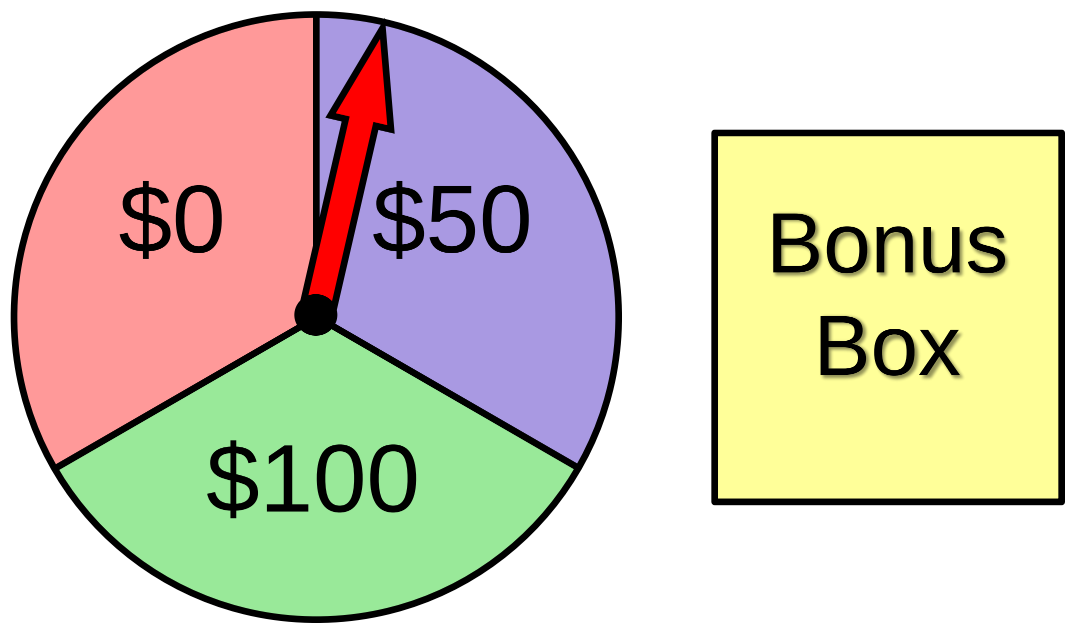
A. $\frac{2}{27}$

B. $\frac{4}{27}$

C. $\frac{2}{9}$

D. $\frac{4}{9}$

E. None of the above



Solution

Answer: B

Justification: Notice that the outcome of the first event (not spinning \$0) does not affect the outcome of the second event (drawing x10). The events are **independent**.

We can use the fundamental counting principle to conclude that the total number of outcomes of both events being executed one after the other is the product of the number of outcomes of each event. The probability of not spinning \$0 *and* drawing a x10 is:

$$P(\text{Not \$0 and x10}) = P(\text{Not \$0}) \cdot P(\text{x10}) = \frac{2}{3} \cdot \frac{2}{9} = \frac{4}{27}$$

Two events are **independent** if and only if:

$$P(\text{A and B}) = P(\text{A}) \cdot P(\text{B})$$

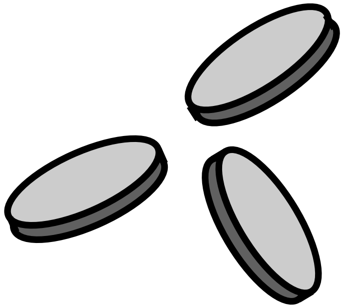
Events VI

A coin is flipped three times. Consider the following 2 events:

A = {The first two coins land heads}

B = {The third coin lands heads}

Are the two events independent or dependent?



A. Independent

B. Dependent

Press for hint

Determine if $P(\text{A and B}) = P(\text{A}) \cdot P(\text{B})$

Solution

Answer: A

Justification: Recall that the sample space of flipping 3 coins is:

S = {TTT, TTH, THT, HTT, THH, HTH, HHT, HHH}

The outcomes of “the first two coins land heads ” are:

A = {HHT, HHH} $P(\text{A}) = \frac{2}{8} = \frac{1}{4}$

The outcomes of “the third coin lands heads” are:

B= {TTH, THH, HTH, HHH} $P(\text{B}) = \frac{4}{8} = \frac{1}{2}$

The outcomes of both A and B are:

(A and B) = {HHH} $P(\text{A and B}) = \frac{1}{8}$ $P(\text{A}) \cdot P(\text{B}) = \frac{1}{4} \cdot \frac{1}{2} = \frac{1}{8}$

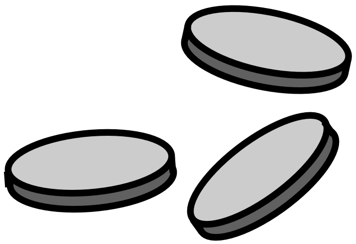
Since *P*(A and B) = *P*(A) · *P*(B), the events are independent. *The outcome of the first two coin flips does not affect the outcome of the third.*

Events VII

A coin is flipped three times. Consider the following 2 events:

A = {The second coin lands heads}

B = {Two heads are landed back to back}



Are the two events independent or dependent?

A. Independent

B. Dependent

Press for hint 

Determine if $P(\text{A and B}) = P(\text{A}) \cdot P(\text{B})$

Solution

Answer: B

Justification: Recall that the sample space of flipping 3 coins is:

S = {TTT, TTH, THT, HTT, THH, HTH, HHT, HHH}

The outcomes of “the second coin land heads ” are:

A = {THT, THH, HHT, HHH} $P(\text{A}) = \frac{4}{8} = \frac{1}{2}$

The outcomes of “two heads are landed back to back” are:

B= {HHT, THH, HHH} $P(\text{B}) = \frac{3}{8}$

The outcomes of both A and B are:

(A and B) = {HHT, THH, HHH} $P(\text{A and B}) = \frac{3}{8}$

Since P(A and B) ≠ P(A) · P(B), the events are dependent.

Solution Continued

Answer: B

Justification: When two events are dependant, the occurrence of one affects the occurrence of the other. In this coin example, in order to land two heads back to back out of three, the second toss must land heads.

A = {The second coin lands heads} = {THT, THH, HHT, HHH}

B = {Two heads are landed back to back} = {HHT, THH, HHH}

(A and B) = {HHT, THH, HHH}

Notice that there is no difference between the set B, and set (A and B). If we know for a fact that the second coin will land heads, there is a much higher probability that we will land two heads back to back.

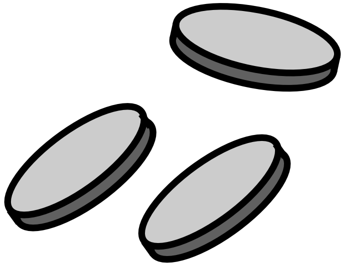
Try the set on conditional probability to learn about dependent events.

Events VIII

A complement of an event A is the set of all outcomes in a sample space that are not in A.

Consider flipping three coins. Which one of the following describes the complement of the event: “Landing at least 1 head?”

A. Landing at least 1 tails
B. Landing at most 1 heads
C. Landing exactly 1 heads
D. Landing all tails
E. Landing all heads



Solution

Answer: D

Justification: Recall that the sample space of the experiment is:

S = {TTT, TTH, THT, HTT, THH, HTH, HHT, HHH}

Event E (getting at least 1 heads) is a subset of this sample space consisting of: E = {TTH, THT, HTT, THH, HTH, HHT, HHH}

The complement of the event E, denoted by $\overline{E}$, is the set of outcomes in S that are not in E.

$$S = \{\underbrace{TTT}_{\overline{E}},\ \underbrace{TTH, THT, HTT, THH, HTH, HHT, HHH}_{E}\} \qquad \overline{E} = \{ TTT \}$$

This set is best described by “Landing all tails.” Notice that the complement of “Landing all tails” is not “Landing all heads.”

Events IX

A coin is flipped 10 times.

What is the probability of landing at least 1 heads?

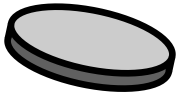
A. $P(\text{At least 1 heads}) = \frac{1}{2^{10}}$

B. $P(\text{At least 1 heads}) = 1 - \frac{1}{2^{10}}$

C. $P(\text{At least 1 heads}) = \frac{1}{2^{9}}$

D. $P(\text{At least 1 heads}) = 1 - \frac{1}{2^{9}}$

E. $P(\text{At least 1 heads}) = \frac{1}{2^{9}} + \frac{1}{2^{8}} + \ldots + \frac{1}{2^{2}} + \frac{1}{2^{1}}$



×10

Solution

Answer: B

Justification: Getting at least 1 heads means landing anywhere between 1 and 10 heads after the 10 coins are flipped. The complement event is getting no heads, or getting all tails.

There is only 1 way to get all tails. There are 2^{10} total outcomes after flipping 10 coins, so the probability of getting all tails is

$$P(\text{All tails}) = \frac{1}{2^{10}}$$

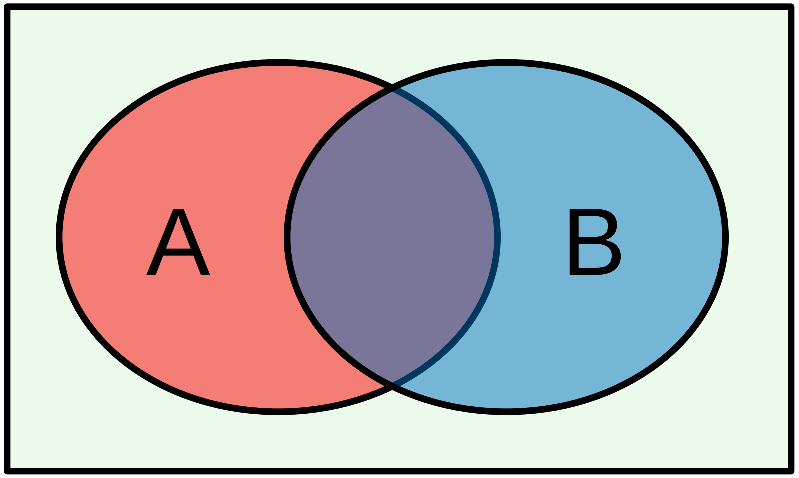
The sum of the probabilities of an event and its complement must be 1. This can be used to determine the probability of getting 1 to 10 heads:

$$P(\text{At least 1 heads}) = 1 - P(\text{All tails}) = 1 - \frac{1}{2^{10}}$$

Events X

Two events are mutually exclusive if they cannot occur at the same time. If events A and B are not mutually exclusive, which one of the following equals *P*(A or B)?

Note: Outcomes where A and B both occur are included in “A or B”



Not mutually exclusive

A. $P(\text{A or B}) = P(\text{A}) + P(\text{B})$

B. $P(\text{A or B}) = P(\text{A}) \cdot P(\text{B})$

C. $P(\text{A or B}) = P(\text{A}) + P(\text{B}) - P(\text{A and B})$

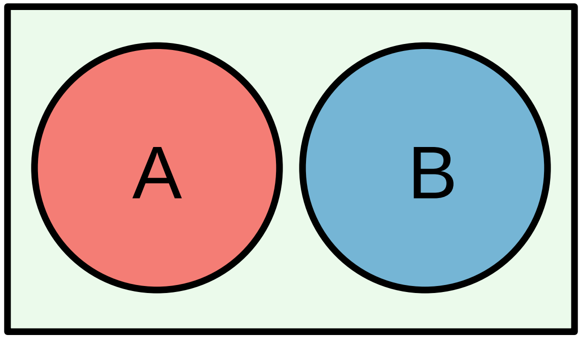
D. $P(\text{A or B}) = P(\text{A}) + P(\text{B}) + P(\text{A and B})$

E. $P(\text{A or B}) = P(\text{A and B}) - P(\text{A}) - P(\text{B})$

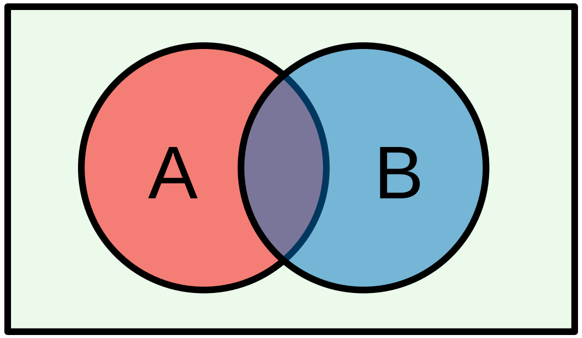
Solution

Answer: C

Justification: P(A or B) is represented by the fraction of the total area covered by the two circles A and B.



Mutually exclusive



Not mutually exclusive

Two events that are not mutually exclusive can both happen at the same time. Adding P(A) and P(B) includes the probability of P(A and B) (when both events happen at the same time) twice.

Therefore, $P(\text{A or B}) = P(\text{A}) + P(\text{B}) - P(\text{A and B})$

Events XI

Consider the following statistics taken from a survey of all grade 12 students:

- 80% of all grade 12 students take Math
- 40% of all grade 12 students take Physics
- 30% of all grade 12 student take both Math and Physics

What percent of students take Physics or Math (or both)?

A. 120%
B. 100%
C. 90%
D. 87%
E. Anything between 80% and 100%

Solution

Answer: C

Justification: Taking Math and taking Physics are not mutually exclusive because the probability that a student takes both Math and Physics is not zero.

$$
\begin{aligned}
P(\text{Physics or Math}) &= P(\text{Physics}) + P(\text{Math}) - P(\text{Physics and Math}) \\
&= 40\% + 80\% - 30\% \\
&= 90\%
\end{aligned}
$$

Note that it is sometimes unclear whether “A OR B” includes the probability that both occur. If we want the probability that “A OR B occur, but not both,” this is known as the “exclusive OR”

Events XII

Consider the following statistics taken from a survey of all grade 12 students:

- 80% of all grade 12 students take Math
- 40% of all grade 12 students take Physics
- 30% of all grade 12 student take both Math and Physics

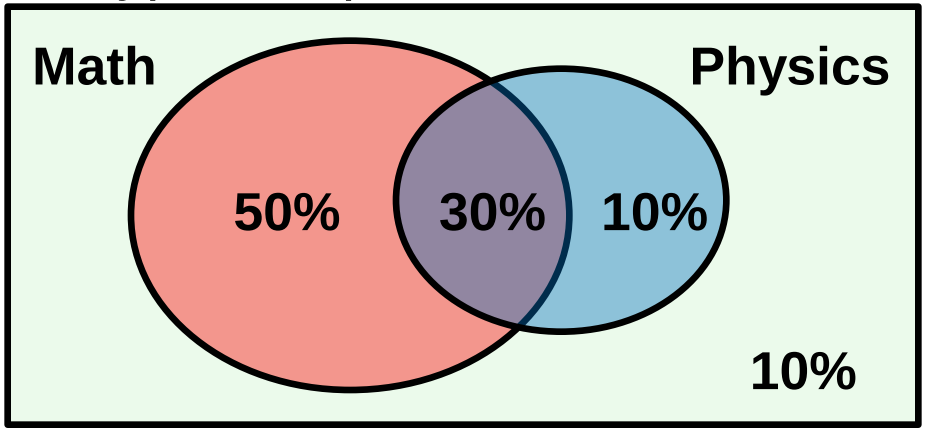
What percent of students take Physics but not Math?

A. 0%
B. 10%
C. 20%
D. 30%
E. Anything between 0% and 20%

Solution

Answer: B

Justification: These types of questions are best solved by considering a Venn-diagram:



30% of all student take Math and Physics. In order for the statistic that 40% of all students take Physics to be true, 10% of all students must take Math and not Physics. This is because

$$P(\text{Physics}) = P(\text{Physics and Math}) + P(\text{Physics and not Math})$$

$$P(\text{Physics and not Math}) = 40\% - 30\% = 10\%$$

Summary

Independent Events:

$$P(\text{A and B}) = P(\text{A}) \cdot P(\text{B})$$

Dependent Events:

$$P(\text{A and B}) \neq P(\text{A}) \cdot P(\text{B})$$

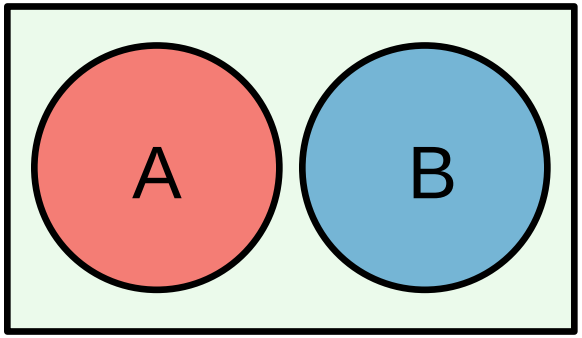
Complement Events:

$$P(\overline{\text{A}}) = 1 - P(\text{A})$$

Mutually Exclusive Events:

$$P(\text{A and B}) = 0$$

$$P(\text{A or B}) = P(\text{A}) + P(\text{B})$$



Not Mutually Exclusive Events:

$$P(\text{A and B}) \neq 0$$

$$P(\text{A or B}) = P(\text{A}) + P(\text{B}) - P(\text{A and B})$$