a place of mind

FACULTY OF EDUCATION

Department of
Curriculum and Pedagogy

Mathematics
Arithmetic Sequences

Science and Mathematics
Education Research Group

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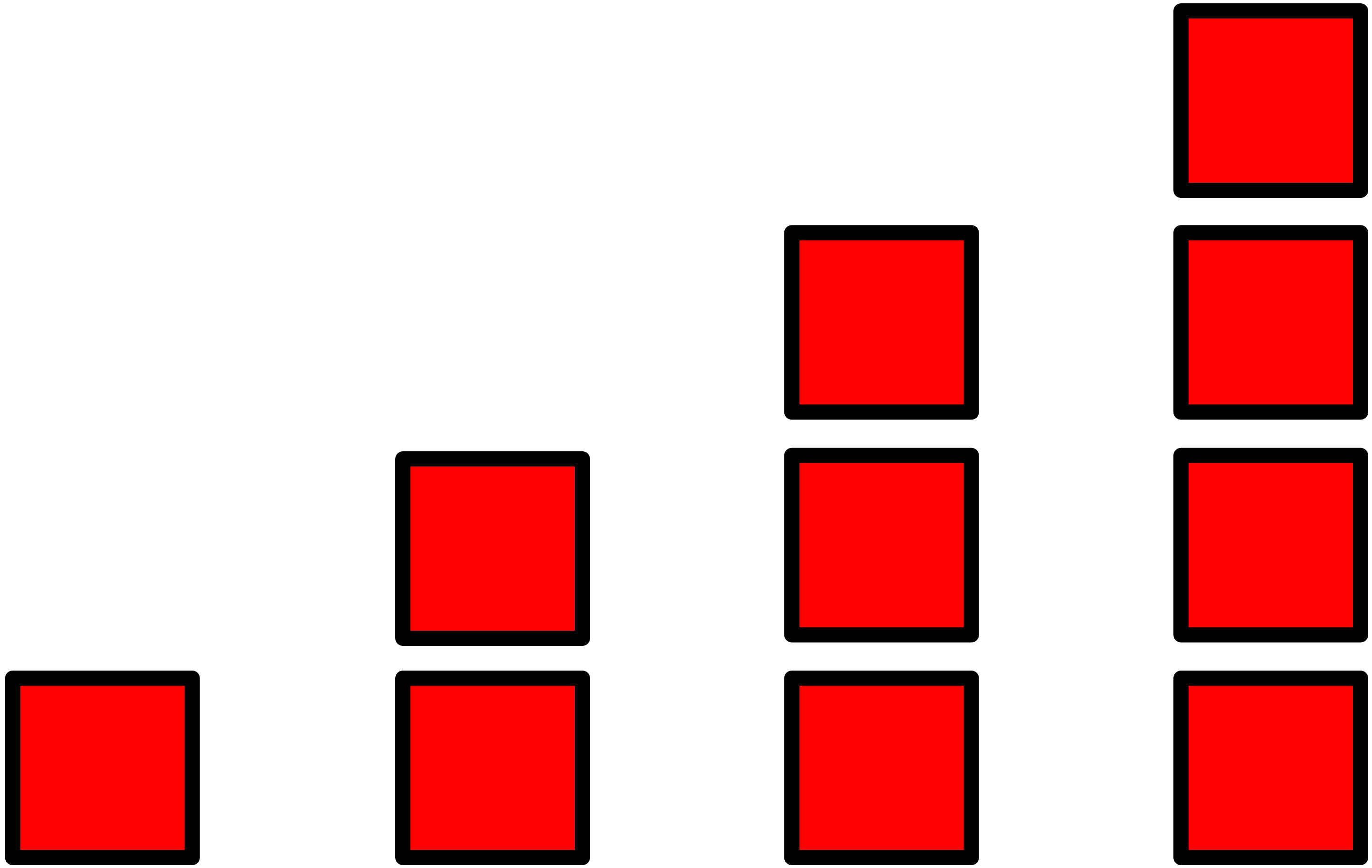
Arithmetic Sequences



Arithmetic Sequences I

Consider the following sequence of numbers:

2, 4, 6, 8, 10, ...

The first 5 terms are shown. What is the 8th term in the arithmetic sequence?

A. 14

B. 16

C. 18

D. 20

E. 22

Solution

Answer: B

Justification: The sequence is called an arithmetic sequence because the difference between any two consecutive terms is 2 (for example 6 – 4 = 2). This is known as the *common difference*. The next term in the sequence can be found by adding the common difference to the last term:

+2 +2 +2

2, 4, 6, 8, 10, 12, 14, 16

↑ *5^{th} term* (10) ↑ *8^{th} term* (16)

Arithmetic Sequences II

Consider the following sequence of numbers:

$a_1, \quad a_2, \quad a_3, \quad a_4, \quad a_5, \ldots$

where a_n is the n^{th} term of the sequence. The common difference between two consecutive terms is *d*. What is a_8, in terms of a_5 and d?

A. $a_8 = a_5 + 3d$

B. $a_8 = a_5 + 3a_1$

C. $a_8 = a_5 + 8d$

D. $a_8 = a_5 + 8a_1$

E. Cannot be determined

Solution

Answer: A

Justification: The next term in the sequence can be found by adding the common difference to the last term:

$$a_1, \quad a_2, \quad a_3, \quad a_4, \quad a_5, \overset{+d}{\quad} a_6, \overset{+d}{\quad} a_7, \overset{+d}{\quad} a_8$$

Only 3 times the common difference has to be added to the 5th term to reach the 8th term.

$$a_8 = a_5 + d + d + d = a_5 + 3d$$

Notice that the first term does not need to be known. As we will see in later questions, it will be helpful to be able to express terms of a sequence with respect to the first term.

Arithmetic Sequences III

Consider the following sequence of numbers:

a_1, a_2, a_3, a_4, a_5, ...

where a_n is the n^{th} term of the sequence. The common difference between two consecutive terms is *d*. What is a_8, in terms of a_1 and d?

A. $a_8 = 8a_1$

B. $a_8 = a_1 + 6d$

C. $a_8 = a_1 + 7d$

D. $a_8 = a_1 + 8d$

E. Cannot be determined

Solution

Answer: C

Justification: The next term in the sequence can be found by adding the common difference to the previous term. Starting at the first term, the common difference must be added 7 times to reach the 8th term:

$$a_1, \overset{+d}{} a_2, \overset{+d}{} a_3, \overset{+d}{} a_4, \overset{+d}{} a_5, \overset{+d}{} a_6, \overset{+d}{} a_7, \overset{+d}{} a_8$$

$$a_8 = a_1 + 7d$$

Note how we do not add 8 times the common difference to reach the 8th term if we are starting at the first term.

Arithmetic Sequences IV

Consider the following sequence of numbers:

$a_1, \quad a_2, \quad a_3, \quad a_4, \quad a_5, \ldots$

where a_n is the n^{th} term of the sequence. The common difference between two consecutive terms is *d*. What is a_n in terms of a_1 and n?

A. $a_n = a_1 + (n)a_1$

B. $a_n = a_1 + (n-1)a_1$

C. $a_n = a_1 + (n)d$

D. $a_n = a_1 + (n-1)d$

E. Cannot be determined

Solution

Answer: D

Justification: Consider the value of the first few terms:

$$a_1 = a_1 + 0d$$
$$a_2 = a_1 + 1d$$
$$a_3 = a_1 + 2d$$
$$a_4 = a_1 + 3d$$
$$\vdots$$
$$a_n = a_1 + (n-1)d$$

Notice that the common difference is added to a_1 (n-1) times, not n times. This is because the common difference is not added to a_1 to get the first term. Also note that the first term remains fixed and we do not add multiples of it to find later terms.

Arithmetic Sequences V

Consider the following arithmetic sequence:

___, ___, ___, 6, 1, ...

What is the 21st term in the sequence?

A. $a_{21} = 6 + 20(5)$
B. $a_{21} = 21 + 20(5)$
C. $a_{21} = 21 + 21(5)$
D. $a_{21} = 21 – 20(5)$
E. $a_{21} = 21 – 21(5)$

Hint: Find the value of the common difference and the first term.

$$a_n = a_1 + (n-1)d$$

Press for hint

Solution

Answer: D

Justification: The common difference is

$$d = a_5 – a_4 = 1 – 6 = -5.$$

Subtracting the common difference from a_n gives a_{n-1}. This gives $a_1 = 21$. Using the formula, $a_n = a_1 + (n-1)d$, we find that:

$$a_{21} = 21 + (21-1)(-5) = 21 – 20(5) = -79$$

Arithmetic Sequences VI

How many numbers are there between 23 and 1023 inclusive (including the numbers 23 and 1023)?

A. 998
B. 999
C. 1000
D. 1001
E. 1002

Hint: Consider an arithmetic sequence with $a_1 = 23$, $a_n = 1023$, and $d = 1$

$$a_n = a_1 + (n-1)d$$

Press for hint

Solution

Answer: D

Justification: The answer is not just 1023 – 23 = 1000. Imagine if we wanted to find the number of terms between 1 and 10. The formula above will give 10 – 1 = 9, which is incorrect.

Consider an arithmetic sequence with $a_1 = 23$, and $a_n = 1023$. The common difference (d) for consecutive numbers is 1. Solving for n, we can find the term number of 1023:

$$a_n = a_1 + (n-1)d$$

$$1023 = 23 + (n-1)1$$

$$n - 1 = 1023 - 23$$

$$n = 1001$$

Since 1023 is the 1001^{th} term in the sequence starting at 23, there are 1001 numbers between 23 and 1023.

Arithmetic Sequences VII

In a particular arithmetic sequence:

$a_{19} = 50$, $a_{30} = 80$

What is the common difference of this sequence?

A. $d = \frac{30}{9}$

B. $d = \frac{30}{10}$

C. $d = \frac{30}{11}$

D. $d = \frac{30}{12}$

E. None of the above

Solution

Answer: D

Justification:

(Method 1):
To get to a_{30} from a_{19}, 11 times the common difference must be added to a_{19}:

$$a_{30} = a_{19} + 11d$$

$$a_{30} - a_{19} = 11d$$

$$30 = 11d$$

$$d = \frac{30}{11}$$

(Method 2):
Using the formulas, a_{19} and a_{30} in terms of a_1 is given by:

$a_{19} = a_1 + 18d$

$a_{30} = a_1 + 29d$

Subtracting a_{30} from a_{19} gives:

$$a_{30} - a_{19} = 11d$$

$$30 = 11d$$

$$d = \frac{30}{11}$$

Arithmetic Sequences VIII

The statements A through E shown below each describe an arithmetic sequence. In which of the arithmetic sequences is the value of a_{10} the largest?

A. $a_1 = 10$; $d = 2$

B. $a_1 = 15$; $d = -3$

C. $a_{11} = 30$; $a_{12} = 20$

D. $a_{20} = 40$; $d = 2$

E. $a_{20} = 40$; $d = -3$

Solution

Answer: E

Justification: It is easy to calculate a_{10} in sequence A since a_1 and d are given: $a_{10} = 10 + 9(2) = 28$.

Sequence B begins at 15, but the common difference is negative, so all terms in statement B are less than 15.

In sequence C, we can see that the common difference is 10 and $a_{10} = 40$ by inspection.

In sequence D, in order to get to a_{10} from a_{20}, we must count down by 2 starting at 40. a_{10} is clearly smaller than 40.

In order to get to a_{10} in sequence E, we must count up by 3 starting at 40 since the common difference is negative. a_{10} in sequence E the largest.