a place of mind

FACULTY OF EDUCATION

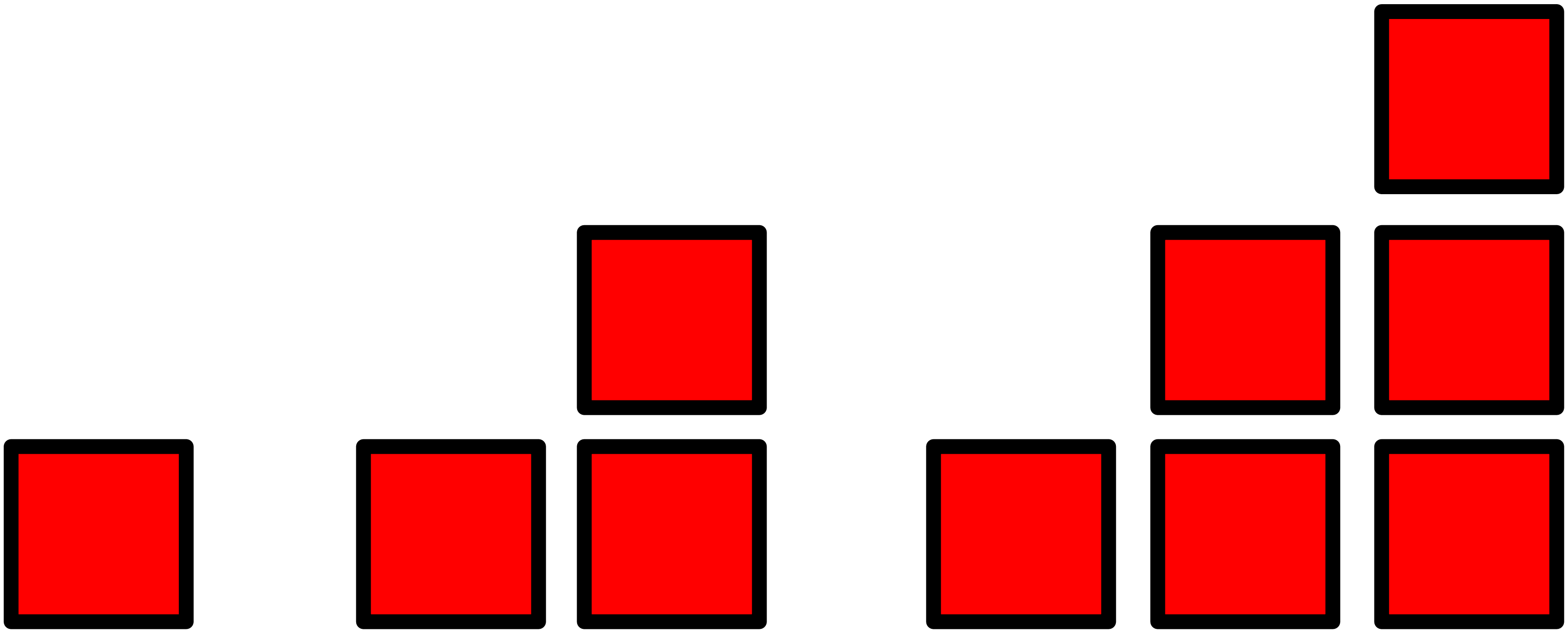
Department of
Curriculum and Pedagogy

Mathematics
Arithmetic Series

Science and Mathematics
Education Research Group

Supported by UBC Teaching and Learning Enhancement Fund 2012-2013

Arithmetic Series



Arithmetic Series I

Suppose we want to determine the sum of the terms of the following sequence:

$$S = 1 + 2 + 3 + 4 + 5 + 6 + 7 + 8 + 9 + 10$$

Which if of the following correctly expresses this sum?

A. $S = 5(5) = 25$

B. $S = 5(10) = 50$

C. $S = 5(11) = 55$

D. $S = 10(10) = 100$

E. $S = 10(11) = 110$

Press for hint

Consider grouping the terms as shown:

$$S = (1+10) + (2+9) + (3+8) + (4+7) + (5+6)$$

Solution

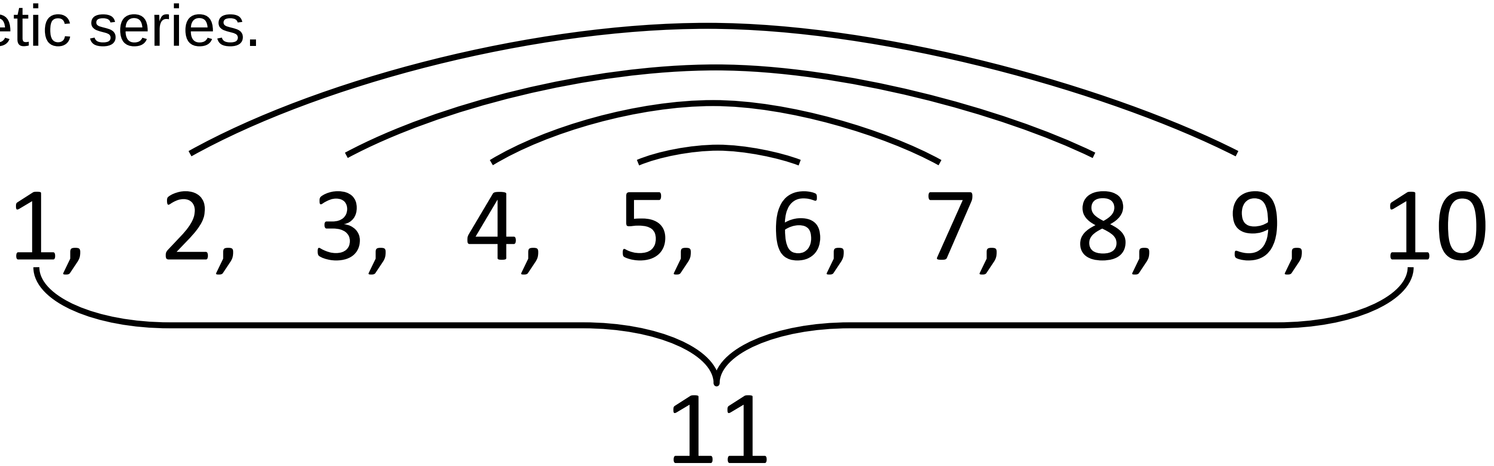
Answer: C

Justification: By grouping the numbers as shown below, the sum is always 11.

S = (1+10) + (2+9) + (3+8) + (4+7) + (5+6)

S = 11 + 11 + 11 + 11 + 11 = 5(11) = 55

Note: The sum of the terms of an arithmetic sequence is known as an arithmetic series.



Arithmetic Series II

Consider the sequences from 1 to 10 and 10 to 1. The terms of these two opposite sequences are added together:

First Term | Last Term

	1	2	3	4	5	6	7	8	9	10
+	10	9	8	7	6	5	4	3	2	1
	11	11	11	11	11	11	11	11	11	11

What is the sum of the numbers from 1 to 10?

A. S = 5(5)

B. S = 5(10)

C. S = 5(11)

D. S = 10(10)

E. S = 10(11)

Solution

Answer: C

Justification: Let the sum between 1 to 10 (or 10 to 1) be S.

$$S = 1 + 2 + 3 + 4 + 5 + 6 + 7 + 8 + 9 + 10$$

$$S = 10 + 9 + 8 + 7 + 6 + 5 + 4 + 3 + 2 + 1$$

The table showed that if we add these together, we will get:

$$2S = 11(10)$$

$$S = 11(5)$$

Notice that since we add the numbers from 1 to 10 to 10 to 1, we have twice the sum of the numbers from 1 to 10. Dividing the result by two gives that the sum of the numbers from 1 to 10 is 55.

This method will be used to derive a general formula for finding the sum of terms in an arithmetic sequence.

Arithmetic Series III

Consider the following two arithmetic sequences with n terms:

	First Term				n^{th} Term
Sequence 1:	a_1	a_1+d	...	$a_1+(n-2)d$	$a_1+(n-1)d$
Sequence 2:	$a_1+(n-1)d$	$a_1+(n-2)d$	...	a_1+d	a_1
			...		

If the nth term of sequence 1 is added to the nth term of sequence 2, is the sum the same for all n? If so, what do the pairs add up to?

A. The terms cannot be paired to give the same sum

B. $2a_1 + (n-1)d$

C. $2a_1 + 2(n-1)d$

D. $2a_1 + (n)d$

E. $2a_1 + 2(n)d$

Solution

Answer: B

Justification:

	First Term				n^{th} Term
Sequence 1:	a_1	a_1+d	...	$a_1+(n-2)d$	$a_1+(n-1)d$
Sequence 2:	$a_1+(n-1)d$	$a_1+(n-2)d$	...	a_1+d	a_1
	$2a_1+(n-1)d$	$2a_1+(n-1)d$	...	$2a_1+(n-1)d$	$2a_1+(n-1)d$

When the first terms are added together, we have

$$a_1+a_1+(n-1)d = 2a_1+(n-1)d.$$

The next term in sequence 1 is the previous term plus d.

The next term in sequence 2 is the previous term minus d.

Therefore, the sum of the second terms will also be $2a_1+(n-1)d$. Every pair of terms will have the same sum, $2a_1+(n-1)d$.

Arithmetic Series IV

	First Term				n^{th} Term
Sequence 1:	a_1	a_1+d	...	$a_1+(n-2)d$	$a_1+(n-1)d$
Sequence 2:	$a_1+(n-1)d$	$a_1+(n-2)d$	...	a_1+d	a_1
	$2a_1+(n-1)d$	$2a_1+(n-1)d$	...	$2a_1+(n-1)d$	$2a_1+(n-1)d$

Let the sum of the terms in sequence 1 (or sequence 2) be S. Which of the following correctly expresses S in terms of the number of terms n, the first term a_1, and the common difference d?

A. $S=\frac{(n-1)}{2}(2a_1+(n-1)d)$

B. $S=\frac{n}{2}(2a_1+(n-1)d)$

C. $S=\frac{(n+1)}{2}(2a_1+(n-1)d)$

D. $S=n\cdot(2a_1+(n-1)d)$

E. $S=2n\cdot(2a_1+(n-1)d)$

Solution

Answer: B

Justification: From the last question, we learned that every pair of terms have the same sum:

$S =$	a_1	+	a_1+d	+ ...	+	$a_1+(n-2)d$	+	$a_1+(n-1)d$
$S =$	$a_1+(n-1)d$	+	$a_1+(n-2)d$	+ ...	+	a_1+d	+	a_1
$2S =$	$2a_1+(n-1)d$	+	$2a_1+(n-1)d$	+ ...	+	$2a_1+(n-1)d$	+	$2a_1+(n-1)d$

Since each sequence has n terms, $2a_1+(n-1)d$ occurs n times. We can therefore express the sum S as:

$$2S = n(2a_1 + (n-1)d)$$

$$S = \frac{n}{2}(2a + (n-1)d)$$

Arithmetic Series V

What is the sum of the following arithmetic series?

$$1 + 2 + 3 + ... + 98 + 99 + 100$$

A. 49(101)

B. 50(100)

C. 50(101)

D. 100(100)

E. 100(101)

Solution

Answer: C

Justification: Since we know both the first term and the last term in the formula, we can conclude that:

$2a_1+(n-1)d$ = (first term) + (last term) = 101

There are 100 terms from 1 to 100, so applying the formula with n = 100 gives:

$$S = \frac{n}{2}(2a+(n-1)d)$$

$$S = \frac{100}{2}(101)$$

$$S = 50(101)$$

$$S = 5050$$

Arithmetic Series VI

The statements A through E shown below each describe an arithmetic sequence.

If the first 20 terms of each sequence are added together, which sequence will give the largest sum?

Hint: Find rough estimates for each sum and compare

A. $a_1 = 100$; $a_{11} = 200$

B. $a_1 = 100$; $a_{21} = 200$

C. $a_1 = 100$; $a_{101} = 200$

D. $a_1 = 200$; $a_{11} = 100$

E. $a_1 = 200$; $a_{21} = 100$

Solution

Answer: A

Justification: The largest sum will be the sequence with the largest $a_1 + a_{20}$ since each series has the same number of terms.

Sequence A, B and C all have the same first term but a different a_{20}. They all also contain the number 200, but this occurs the earliest in A. Therefore, A must have the largest common difference and the largest a_{20} (≈ 300).

Sequence D and E both have a larger first term than A, but they are both decreasing sequences. The 20^{th} term in sequence D is much smaller than 100, so sum of the first 20 terms of D will be smaller than A.

The 20^{th} term in D will be slightly larger than 100. Compared with A which has $a_1 = 100$ and $a_{20} \approx 300$, we can conclude that the sum of the first 20 terms of A will be the largest.

Arithmetic Series VII

Suppose we know that the sum of the first 100 terms in a sequence is 27300. The sum of the first 101 terms in the same sequence is 27876. Which of the following is true about the arithmetic sequence?

A. $a_{100} = 576$

B. $a_{100} = -576$

C. $a_{101} = 576$

D. $a_{101} = -576$

E. We cannot learn anything about the sequence

Solution

Answer: C

Justification:

The sum of the first 100 terms in a sequence is:

$$S_{100} = a_1 + a_2 + a_3 + a_4 + ... + a_{99} + a_{100} = 27300$$

The sum of the first 101 terms in a sequence is:

$$S_{101} = a_1 + a_2 + a_3 + a_4 + ... + a_{99} + a_{100} + a_{101} = 27876$$

If we subtract S_{100} from S_{101}, nearly all the terms cancel except for a_{101}. Therefore:

$$S_{101} – S_{100} = a_{101} = 27876 – 27300 = 576$$

In general, $S_n – S_{n-1} = a_n$

Arithmetic Series VIII

Compute the following:

$$\log_{10}(1 \cdot 10 \cdot 100 \cdot \quad \ldots \quad \cdot 10^{99} \cdot 10^{100})$$

A. 100

B. 5000

C. 5050

D. 10^{100}

E. 10^{5050}

Hint:

$log_{10}(ab) = log_{10}(a) + log_{10}(b)$

Press for hint

Solution

Answer: C

Justification:

The logarithm can be expanded to:

$$\log 1 + \log 10 + \log 100 + \ldots + \log 10^{99} + \log 10^{100}$$

$$= \log 1 + \log 10 + 2\log 10 + \ldots + 99\log 10 + 100\log 10$$

This is the same as the series:

$$0 + 1 + 2 + 3 + \ldots + 99 + 100$$

$$S_{101} = \frac{101}{2}(0 + 100)$$

$$S_{101} = 5050$$

Reminder: Log Rules

$$\log_{10}(ab) = \log_{10}(a) + \log_{10}(b)$$

$$\log_{10}(a^b) = b \cdot \log_{10}(a)$$

$$\log_{10}(1) = 0, \quad \log_{10}(10) = 1$$

Arithmetic Series IX

Tom must deliver pizza to every floor in a 20 floor building. There is 1 flight of stairs between each floor, starting between the first and the second floor. Once Tom delivers pizza to a floor, he must walk all the way back down to his truck to get more pizza. For example, to deliver pizza to the 5th floor, he goes up 4 flights and down 4 flights of stairs. How many flights of stairs does he have to go up and down to deliver pizza to every floor in the building?

A. 190 flights

B. 200 flights

C. 380 flights

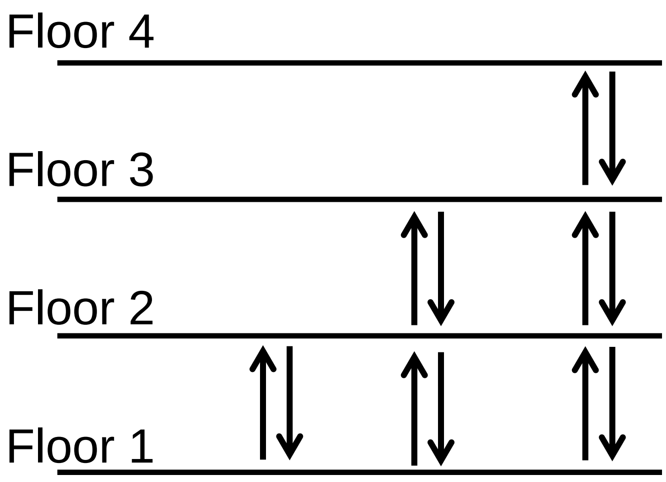
D. 400 flights

E. 0 since Tom takes the elevator

Solution

Answer: C

Justification:



$$S_n = \frac{n}{2}(2a + (n-1)d)$$

The number of stairs to go up and down to each floor is:

Floor 1: 0 stairs
Floor 2: 2 stairs
Floor 3: 4 stairs
Floor 4: 6 stairs

To deliver each every floor, we must compute the sum of the first 20 terms of the sequence with $a_1 = 0$, $d = 2$:

0, 2, 4, 6 ....

$$S_{20} = \frac{20}{2}(2(0) + (20-1)2)$$

$$S_{20} = 10(38) = 380$$