a place of mind

FACULTY OF EDUCATION

Department of
Curriculum and Pedagogy

Mathematics
Geometric Series

Science and Mathematics
Education Research Group

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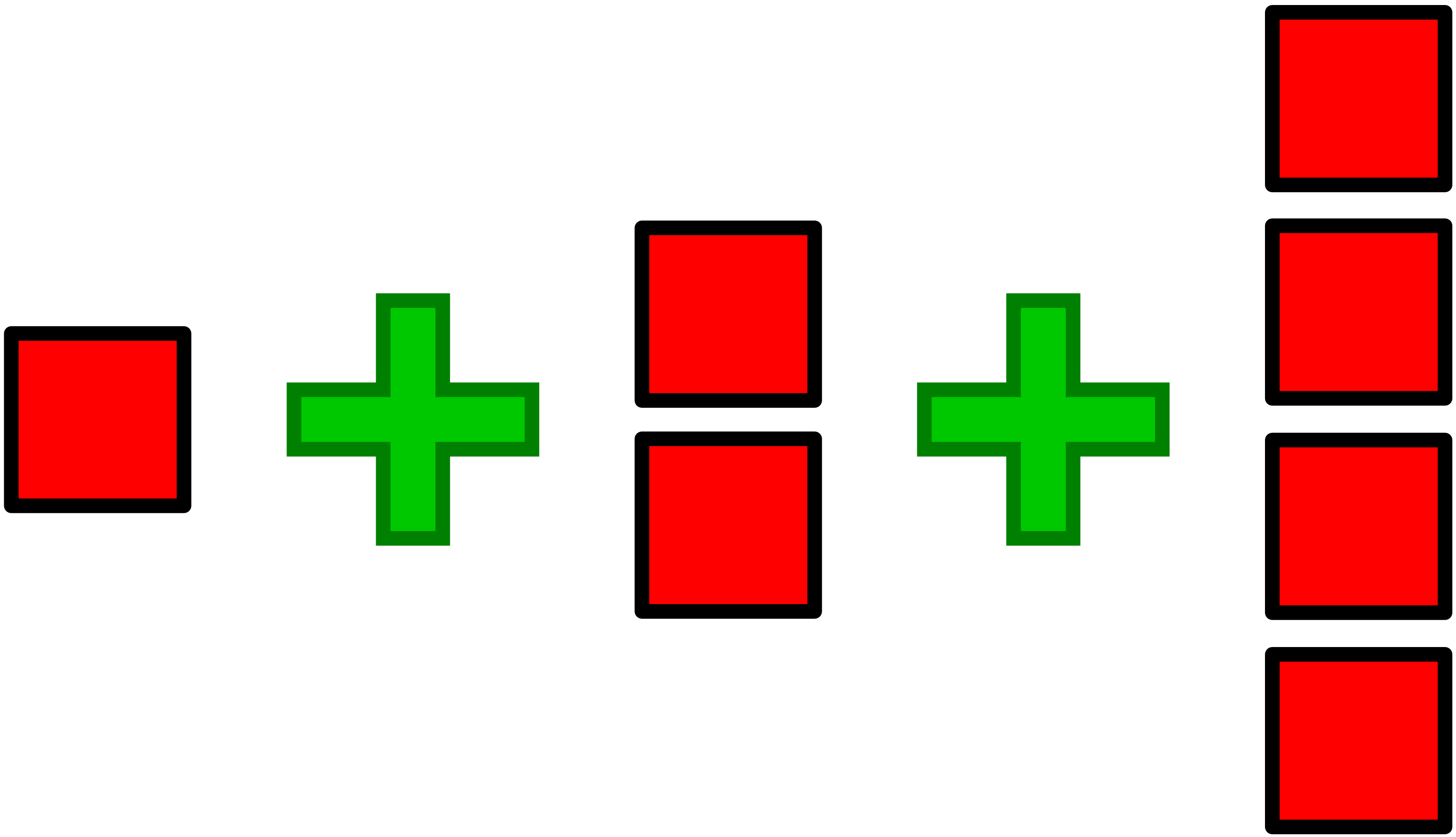
Geometric Series



Geometric Series (Proof) I

Consider adding the first n terms in a geometric sequence:

$$S_n = a_1 + a_1 r + a_1 r^2 + ... + a_1 r^{n-1}$$

Which of the following is the correct value of $r \cdot S_n$?

A. $r \cdot S_n = a_1 r + a_1 r^2 + a_1 r^3 + ... + a_1 r^{n-1}$

B. $r \cdot S_n = a_1 r + a_1 r^2 + a_1 r^3 + ... + a_1 r^{n}$

C. $r \cdot S_n = a_1 r + a_1 r^2 + a_1 r^3 + ... + a_1 r^{n+1}$

D. $r \cdot S_n = a_1 + a_1 r + a_1 r^2 + ... + a_1 r^{n-1}$

E. $r \cdot S_n = a_1 + a_1 r + a_1 r^2 + ... + a_1 r$

Solution

Answer: B

Justification: In order to multiply S_n and r together, each term in S_n is multiplied by r. Using exponent rules, we must add 1 to the exponent on r.

$$a_1 \rightarrow a_1 r$$

$$a_1 r^{n-1} \rightarrow a_1 r^n$$

$$S_n = a_1 + a_1 r + a_1 r^2 + \ldots + a_1 r^{n-1}$$

$$r \cdot S_n = a_1 r + a_1 r^2 + a_1 r^3 + \ldots + a_1 r^n$$

Geometric Series (Proof) II

The two geometric series discovered in the last question are to be subtracted.

$$S_n = a_1 + a_1 r + a_1 r^2 + ... + a_1 r^{n-1}$$

$$r \cdot S_n = a_1 r + a_1 r^2 + a_1 r^3 + ... + a_1 r^n$$

What is the value of $S_n - rS_n$?

A. $a_1 - a_1 r^{n-1}$

B. $a_1 - a_1 r^n$

C. $a_1 + a_1 r^{n-1} - a_1 r^n$

D. $a_1 + a_1 r^n - a_1 r^{n-1}$

E. $1 - r$

Solution

Answer: B

Justification: Notice how most of the terms appear in both of the geometric series. Subtracting the two series will cause these terms to cancel:

$$S_n = a_1 + \cancel{a_1 r} + \cancel{a_1 r^2} + \cancel{a_1 r^3} + \ldots + \cancel{a_1 r^{n-1}}$$
$$r \cdot S_n = \cancel{a_1 r} + \cancel{a_1 r^2} + \cancel{a_1 r^3} + \ldots + \cancel{a_1 r^{n-1}} + a_1 r^n$$

Note that the $a_1 r^{n-1}$ term also cancels. It is the second last term of the second series. After subtracting, all that remains is the first term and last term of the second series.

$$S_n - r \cdot S_n = a_1 - a_1 r^n$$

Geometric Series (Proof) III

From

$$S_n - r \cdot S_n = a_1 - a_1 r^n,$$

how can S_n be isolated to generate a formula that determines the sum of the first n terms in a geometric series?

A. $S_n = \dfrac{a_1 - a_1 r^n}{1 - r}$

B. $S_n = \dfrac{a_1 - a_1 r^n}{r - 1}$

C. $S_n = \dfrac{a_1 - a_1 r^n}{r}$

D. $S_n = a_1 - 2a_1 r^n$

E. $S_n = a_1 - a_1 r^n + rS_n$

Solution

Answer: A

Justification: S_n can be isolated to the left-hand side of the equation as shown:

$$S_n - r \cdot S_n = a_1 - a_1 r^n$$

$$S_n(1-r) = a_1 - a_1 r^n \quad \textit{Factor out } S_n$$

$$\boxed{S_n = \frac{a_1 - a_1 r^n}{1-r}} \quad \textit{Divide by } (1-r)$$

This formula tells us the sum of the first n terms of a geometric series with first term a_1 and common ratio r.

Geometric Series IV

Consider a geometric series with the first term 10 and common ratio 1. If the first 9 terms of this series are added together, what is the final sum?

$$S_n = \frac{a_1 - a_1 r^n}{1-r}$$

A. $S_n = 0$

B. $S_n = 10$

C. $S_n = 90$

D. $S_n = 990$

E. No solution

Solution

Answer: C

Justification: Attempting to substitute r = 1 into the formula for geometric series gives 0 in the denominator. The formula has the restriction that r ≠ 1. However, the sum of a series with r = 1 can be easily found since every term is the same:

$$S_9 = 10 + 10 + 10 + 10 + 10 + 10 + 10 + 10 + 10 = 9(10) = 90$$

The formula for a series when r = 1 can be found using the definition of S_n:

$$S_n = a_1 + a_1 r + a_1 r^2 + ... + a_1 r^{n-1}$$

$$S_n = a_1 + a_1 + a_1 + ... + a_1$$

$$S_n = n a_1$$

Geometric Series V

$$S_n = \frac{a_1 - a_1 r^n}{1 - r}$$

Consider the following geometric series:

$$2^0 + 2^1 + 2^2 + 2^3 + ... + 2^{n-1}$$

Is this sum of this series larger than or equal to 2^n?

A. Larger than 2^n

B. Equal to 2^n

C. Less than 2^n

D. 2^n is larger until n becomes larger than a certain value

E. 2^n is smaller until n becomes larger than a certain value

Solution

Answer: C

Justification: The geometric series has first term 1 and common ratio 2:

$$S_n = 1 + 2 + 4 + 8 + ... + 2^{n-1}$$

$$S_n = \frac{1 - 1(2)^n}{1 - (2)}$$

$$S_n = \frac{1 - 2^n}{-1}$$

$$S_n = 2^n - 1$$

The sum of this geometric series is always one less than 2^n.

Geometric Series VI

What is the sum of the first 10 terms of the following geometric series?

100 + 10 + 1 + 0.1 + ...

$$S_n = \frac{a_1 - a_1 r^n}{1 - r}$$

A. 111.11111111 (11 ones in total)

B. 111.1111111 (10 ones in total)

C. 111.111111 (9 ones in total)

D. 99.99999999 (10 nines in total)

E. 99.9999999 (9 nines in total)

Solution

Answer: B

Justification: This question can be solved using the formula (with $a_1 = 100$, $r = 0.1$, $n = 10$) as shown:

$$S_{10} = \frac{100 - 100(0.1)^{10}}{1-(0.1)} = \frac{99.99999999}{0.9} = 111.1111111$$

However, this is difficult to do without a calculator.

Instead, notice that each term gets added to a different decimal point. Since 10 terms are added together, there should be a total of 10 ones.

$$\begin{array}{rl} & 100. \\ + & 10. \\ + & 1. \\ + & 0.1 \\ + & 0.01 \\ + & \ldots \\ \hline & 111.1111111 \end{array}$$

Geometric Series VII

What is the sum of the series with the following properties?

$a_1 = -1$

$r = -1$

$n = 99$

$$S_n = \frac{a_1 - a_1 r^n}{1 - r}$$

A. $S_{99} = -99$

B. $S_{99} = -1$

C. $S_{99} = 0$

D. $S_{99} = 1$

E. $S_{99} = 99$

Solution

Answer: B

Justification: It is often helpful to write out the first few terms of a series:

$$S_{99} = (-1) + (-1)(-1) + (-1)(-1)^2 + (-1)(-1)^3 + \ldots$$
$$= -1 + 1 - 1 + 1 \ldots$$

By analyzing the pattern, $S_n = 0$ when n is even, and $S_n = -1$ when n is odd. Therefore, the sum of the series when n = 99 is -1.

If the formula is used, the same result is achieved:

$$S_{99} = \frac{-1 - (-1)(-1)^{99}}{1 - (-1)} = \frac{-1 + (-1)}{2} = -1$$

Geometric Series VIII

Starting Jan 1, 2008, Jacob made $10000 a year doing part time work. Each year his salary increases by 5%. How much money will Jacob accumulate by Jan 1, 2020? Assume he is only paid once per year on Dec 31^{st}.

A. The sum of a geometric series with $a_1 = 10000$, $r = 0.05$, $n = 13$
B. The sum of a geometric series with $a_1 = 10000$, $r = 0.05$, $n = 12$
C. The sum of a geometric series with $a_1 = 10000$, $r = 1.05$, $n = 13$
D. The sum of a geometric series with $a_1 = 10000$, $r = 1.05$, $n = 12$
E. This scenario cannot be modeled with a geometric series

Solution

Answer: D

Justification: The question asks for an accumulated amount, so Jacob's earnings each year must be added. Each year Jacob's salary increases by 5%, so his salary is 1.05 times larger than the previous.

The geometric series begins at 10000 with common ratio 1.05. Jacob gets paid 12 times, starting in 2008 and ending in 2019. Jacob is not paid yet for 2020 on Jan 1, 2020. Therefore, only the first 12 terms must be added.

$$S_{12} = \frac{10000 - 10000(1.05)^{12}}{1 - 1.05} = \$159171.27$$

Check: The answer should be greater than 12(10000) = $120000

Geometric Series IX

Consider an equilateral triangle an area of 64 cm^2. A new triangle is drawn inside it by connecting the midpoints of each side as shown. After 5 of these triangles are drawn, they are spread out so that they do not overlap. What is the total area covered by the 5 triangles?

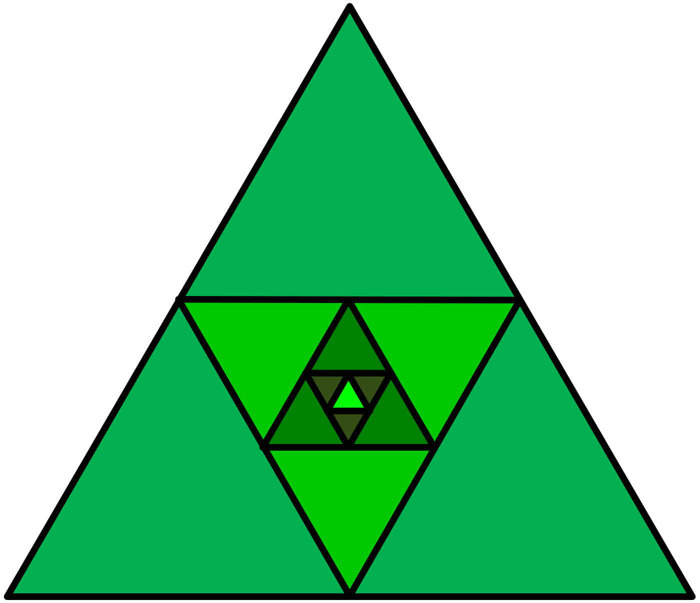
A. 85 cm^2

B. 85.25 cm^2

C. 124 cm^2

D. 132 cm^2

E. 256 cm^2



Solution

Answer: B

Justification: Drawing an equilateral triangle by connecting midpoints creates divides the original triangle into 4 equal size smaller equilateral triangles with half the side length. Each triangle that is drawn is therefore has an area which is 4 times smaller. The total area of all 5 triangles is therefore represented by this geometric series:

$$64 + 16 + 4 + 1 + 0.25 = 85.25$$

Answer B is the only answer with a decimal of 0.25, so the answer can be found quickly without doing a full calculation.

$$S_5 = \frac{64 - 64(0.25)^5}{1 - 0.25} = 85.25$$