Trigonometric Ratios Using Reference Angles
How many different values of $\theta$ between $0^\circ$ and $360^\circ$ are there such that $\sin(\theta) = 0.75$?

A. 0 values of $\theta$
B. 1 value of $\theta$
C. 2 values of $\theta$
D. 3 values of $\theta$
E. 4 values of $\theta$
Solution

Answer: C

Justification: If $\sin(\theta) = 0.75$, then the y-coordinate on the unit circle must be 0.75. This occurs where the line $y = 0.75$ crosses the unit circle.

The diagram shows that the line crosses the unit circle at 2 points, so there are 2 values for $\theta$ where $\sin(\theta) = 0.75$.

This problem set will go over how to find the other angle of $\theta$. 
Consider the 4 points that are 50° from the x-axis. What is the angle θ (the angle to P₂)?

A. 100°
B. 110°
C. 120°
D. 130°
E. 150°
Answer: D

Justification: The diagram above shows that the angle $\theta$ can be calculated by: $\theta = 180^\circ - 50^\circ = 130^\circ$.

The acute angle to the x-axis from $130^\circ$ is $50^\circ$, which is known as the reference angle of $130^\circ$. 
Which of the statements below is true, if any?

A. \( \sin(50^\circ) = \sin(130^\circ) \) and \( \cos(50^\circ) = \cos(130^\circ) \)

B. \( \sin(50^\circ) = -\sin(130^\circ) \) and \( \cos(50^\circ) = \cos(130^\circ) \)

C. \( \sin(50^\circ) = \sin(130^\circ) \) and \( \cos(50^\circ) = -\cos(130^\circ) \)

D. \( \sin(50^\circ) = -\sin(130^\circ) \) and \( \cos(50^\circ) = -\cos(130^\circ) \)

E. None of the above are true
Answer: C

Justification: The y-coordinates of $P_1$ and $P_2$ are the same. Therefore $\sin(50^\circ) = \sin(130^\circ)$.

The x-coordinate of $P_2$ is the same as $P_1$ except negative. Therefore $\cos(50^\circ) = -\cos(130^\circ)$. 
For what value of $\theta$ in the 4$^{th}$ quadrant does $\cos(\theta) = \cos(50^\circ)$?

A. $\theta = 230^\circ$
B. $\theta = 290^\circ$
C. $\theta = 300^\circ$
D. $\theta = 310^\circ$
E. None of the above
Solution

Answer: D

The angle to $P_4$ is $360^\circ - 50^\circ = 310^\circ$ (the reference angle to $310^\circ$ is $50^\circ$). At this point we can see that the $x$-coordinate of $P_1$ and $P_4$ are equal, so:

$$\cos(310^\circ) = \cos(50^\circ)$$
The angle to $P_3$ is $230^\circ$. The reference angle of $230^\circ$ is $50^\circ$.

Which of the following statements is true?

A. $\sin(50^\circ) = \sin(230^\circ)$
B. $\cos(50^\circ) = \cos(230^\circ)$
C. $\tan(50^\circ) = \tan(230^\circ)$
D. A and B are true
E. A, B and C are true
All 3 trigonometric ratios are positive in the first quadrant. The only trigonometric ratio that is positive in the 3\textsuperscript{rd} quadrant is tangent. Only $\tan(50^\circ) = \tan(230^\circ)$ is true.

However, since the $x$ and $y$ coordinates of $P_3$ are negative, we can also conclude that:

$\sin(50^\circ) = -\sin(230^\circ)$

$\cos(50^\circ) = -\cos(230^\circ)$
Quadrant I
\[ \sin(\theta) > 0 \]
\[ \cos(\theta) > 0 \]
\[ \tan(\theta) > 0 \]

Quadrant II
\[ \sin(\theta) > 0 \]
\[ \cos(\theta) < 0 \]
\[ \tan(\theta) < 0 \]

Quadrant III
\[ \sin(\theta) < 0 \]
\[ \cos(\theta) < 0 \]
\[ \tan(\theta) > 0 \]

Quadrant IV
\[ \sin(\theta) < 0 \]
\[ \cos(\theta) > 0 \]
\[ \tan(\theta) < 0 \]
**Summary**

**Quadrant II**
\[ \sin(\theta_r) = \sin(180^\circ - \theta_r) \]

**Quadrant III**
\[ \sin(\theta_r) = \sin(180^\circ - \theta_r) \]

**Quadrant IV**
\[ \tan(\theta_r) = \tan(180^\circ + \theta_r) \]
\[ \cos(\theta_r) = \cos(360^\circ - \theta_r) \]
The value of \( \cos(70^\circ) \) is approximately 0.34. At what other angle does \( \cos(\theta) = 0.34 \), for \( 0^\circ \leq \theta \leq 360^\circ \)?

A. \( \theta = 110^\circ \)  
B. \( \theta = 250^\circ \)  
C. \( \theta = 290^\circ \)  
D. \( \theta = 340^\circ \)  
E. \( \cos(70^\circ) = 0.34 \) for only 1 value of \( \theta \)
**Solution**

**Answer:** C

**Justification:** The value of $\cos(\theta)$ is the same where the line $x = 0.34$ intersects the unit circle (these 2 points have the same x-coordinate).

Cosine is positive in the 1\textsuperscript{st} and 4\textsuperscript{th} quadrants. The angle whose reference angle is $70^\circ$ in the 4\textsuperscript{th} quadrant is $360^\circ - 70^\circ = 290^\circ$.

$\cos(270^\circ) = \cos(70^\circ) = 0.34$

The next questions expect students to be proficient at finding equivalent trigonometric ratios in other quadrants.
Find an angle between 90° and 180° where \( \sin(\theta) = \cos(70°) \).

A. \( \theta = 110° \)
B. \( \theta = 120° \)
C. \( \theta = 150° \)
D. \( \theta = 160° \)
E. No such value of \( \theta \) exists
**Answer:** D

**Justification:** Reflecting the point $P_1$ through the line $y = x$ gives $P_2 = (\sin 70^\circ, \cos 70^\circ)$ by interchanging the $x$ and $y$ coordinates. However, the diagram shows $P_2$ can be written as $(\cos 20^\circ, \sin 20^\circ)$. Equating these two expressions for $P_2$ gives:

$$ (\sin 70^\circ, \cos 70^\circ) = (\cos 20^\circ, \sin 20^\circ). $$

Finally, the equivalent of $\sin(20^\circ)$ in the 2$^{nd}$ quadrant is $\sin(160^\circ)$.

In general: $\cos(\theta) = \sin(90^\circ - \theta)$
Answer: D

Justification: The graphs of sine and cosine are shown below:

Phase shifting the sine graph to the left by 90° (by replacing $\theta$ with $\theta+90°$) gives the cosine graph. This gives us the identity $\cos(\theta) = \sin(\theta+90°)$.

When $\theta=70°$, $\cos(70°) = \sin(160°)$, which agrees with our previous solution.
What is the smallest angle $\theta$ greater than 1000° such that $\sin(\theta) = \sin(255°)$?

A. $\theta = 1005°$
B. $\theta = 1155°$
C. $\theta = 1185°$
D. $\theta = 1335°$
E. No such value of $\theta$ exists
Answer: A

Justification: Adding multiples of 360° to θ does not change the value of sin(θ). So,

\[
\sin(255°) = \sin(615°) = \sin(975°) = \sin(1335°)
\]

However, this is not the smallest angle greater than 1000°. The equivalent of sin(255°) in the 4th quadrant is sin(285°). Adding multiples of 360° to 285° gives:

\[
\sin(285°) = \sin(645°) = \sin(1005°)
\]

Therefore the smallest angle of θ greater than 1000° where sin(θ) = sin(255°) is θ = 1005°.
Find the smallest positive angle $\theta$ where:

$$\tan \theta = \frac{\sin(99^\circ)}{\cos(9^\circ)}$$

A. $\theta = 0^\circ$
B. $\theta = 30^\circ$
C. $\theta = 45^\circ$
D. $\theta = 60^\circ$
E. $\theta = 90^\circ$
Answer: C

Justification: The equivalent of \( \sin(99\degree) \) in the first quadrant is \( \sin(81\degree) \).

Using the same argument from question 7, we can conclude that:

\[
\sin(81\degree) = \cos(9\degree)
\]

Therefore:

\[
\frac{\sin(99\degree)}{\cos(9\degree)} = 1 = \tan(45\degree)
\]