a place of mind

FACULTY OF EDUCATION

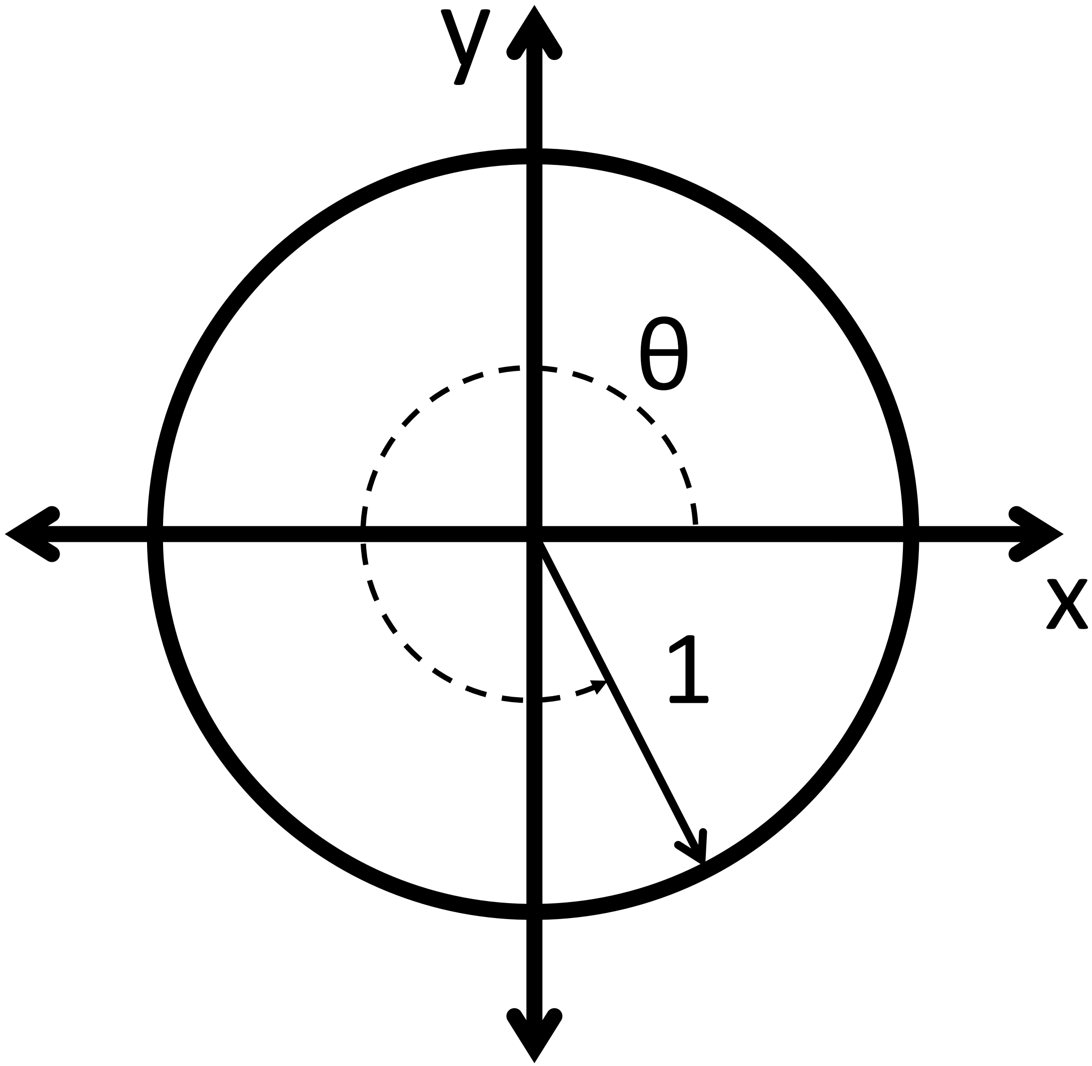
Department of
Curriculum and Pedagogy

Mathematics
Trigonometry: Unit Circle

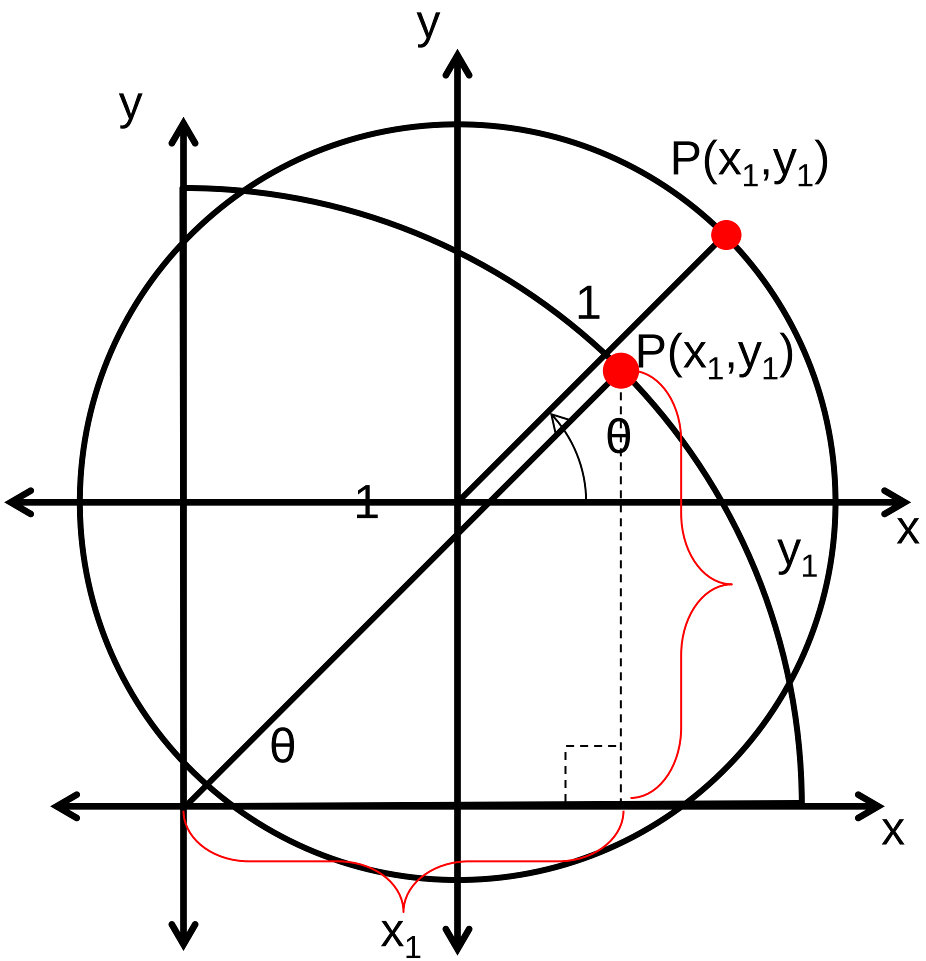
Science and Mathematics
Education Research Group

Supported by UBC Teaching and Learning Enhancement Fund 2012-2013

The Unit Circle



The Unit Circle I



A circle with radius 1 is drawn with its center through the origin of a coordinate plane. Consider an arbitrary point P on the circle. What are the coordinates of P in terms of the angle θ?

Press for hint

A. $P(\theta,\ \theta)$

B. $P(\sin\theta,\ \cos\theta)$

C. $P(\cos\theta,\ \sin\theta)$

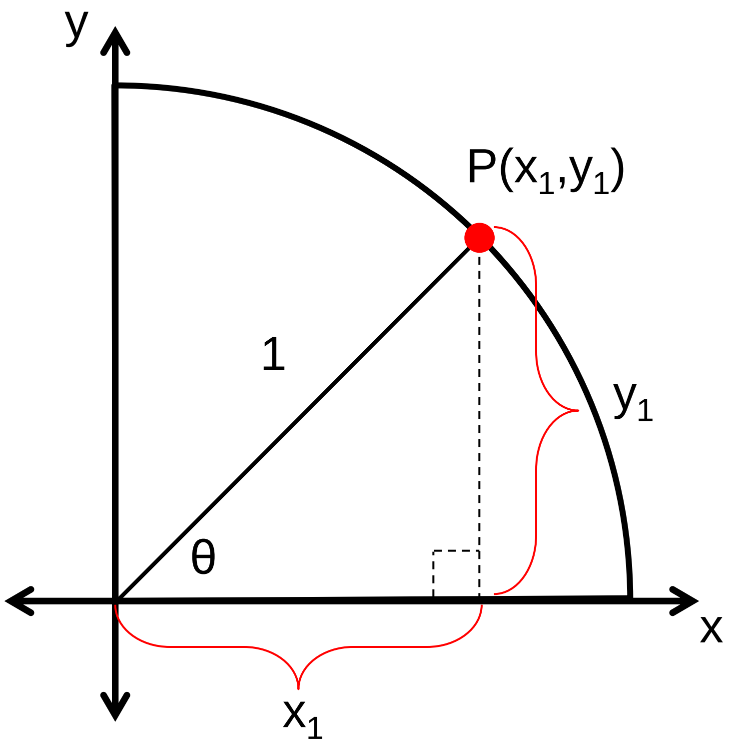
D. $P(\sin^{-1}\theta,\ \cos^{-1}\theta)$

E. $P(\cos^{-1}\theta,\ \sin^{-1}\theta)$

Solution

Answer: C

Justification: Draw a right triangle by connecting the origin to point P, and drawing a perpendicular line from P to the x-axis. This triangle has side lengths x_1, y_1, and hypotenuse 1.



The trigonometric ratios sine and cosine for this triangle are:

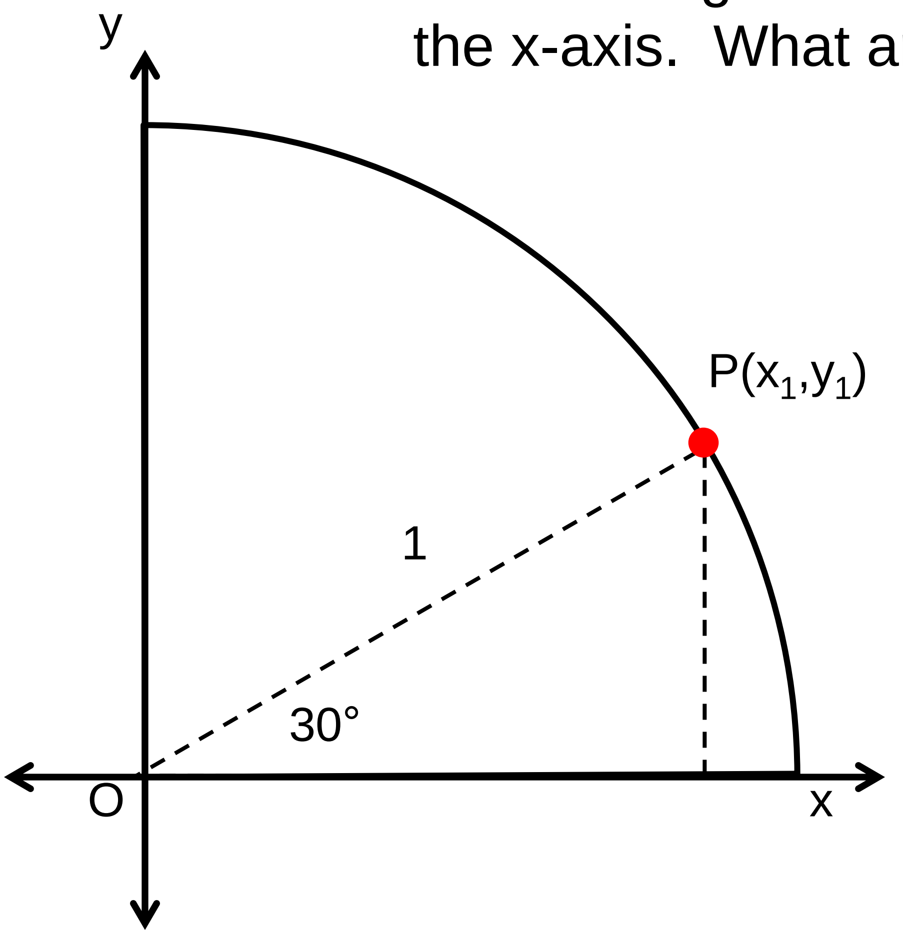
$$\cos(\theta) = \frac{x_1}{1} \quad \Rightarrow \quad x_1 = \cos(\theta)$$

$$\sin(\theta) = \frac{y_1}{1} \quad \Rightarrow \quad y_1 = \sin(\theta)$$

Therefore, the point P has the coordinates (cos θ, sin θ).

The Unit Circle II

The line segment OP makes a 30° angle with the x-axis. What are the coordinates of P?



Hint: What are the lengths of the sides of the triangle?

A. P (2, 1)

B. P ($\sqrt{3}$, 2)

C. P (2, $\sqrt{3}$)

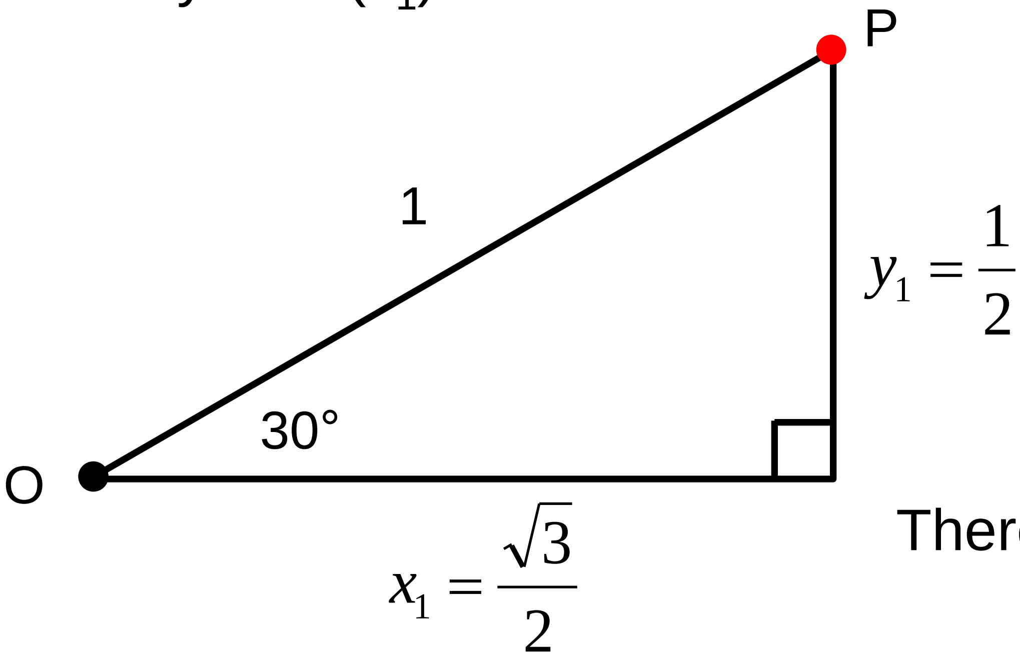
D. P ($\frac{1}{2}$, $\frac{\sqrt{3}}{2}$)

E. P ($\frac{\sqrt{3}}{2}$, $\frac{1}{2}$)

Solution

Answer: E

Justification: The sides of the 30-60-90 triangle give the distance from P to the x-axis (y_1) and the distance from P to the y-axis (x_1).

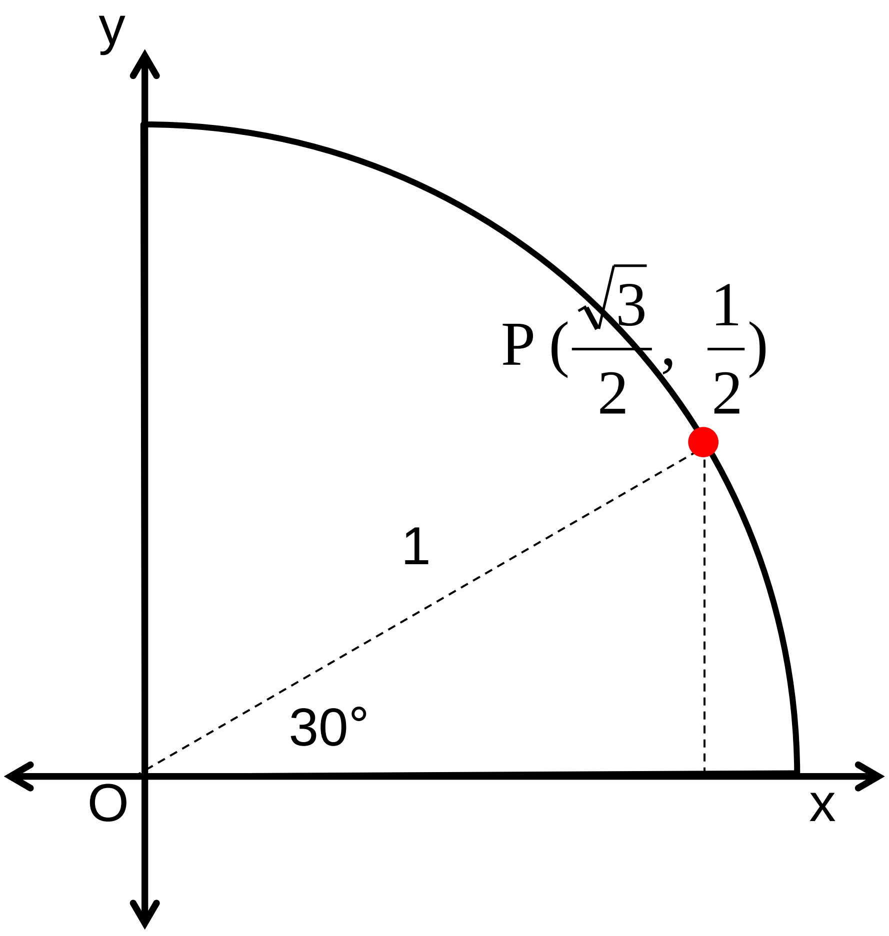


Therefore, the coordinates of P are:

$$(x_1, y_1) = (\frac{\sqrt{3}}{2}, \ \frac{1}{2})$$

The Unit Circle III

What are the exact values of sin(30°) and cos(30°)?



A. $\sin(30°) = \frac{1}{2}, \quad \cos(30°) = \frac{\sqrt{3}}{2}$

B. $\sin(30°) = \frac{\sqrt{3}}{2}, \quad \cos(30°) = \frac{1}{2}$

C. $\sin(30°) = 1, \quad \cos(30°) = \sqrt{3}$

D. $\sin(30°) = \sqrt{3}, \quad \cos(30°) = 1$

E. Cannot be done without a calculator

Solution

Answer: A

Justification: From question 1 we learned that the x-coordinate of P is cos(θ) and the y-coordinate is sin(θ). In question 2, we found the x and y coordinates of P when θ = 30° using the 30-60-90 triangle. Therefore, we have two equivalent expressions for the coordinates of P:

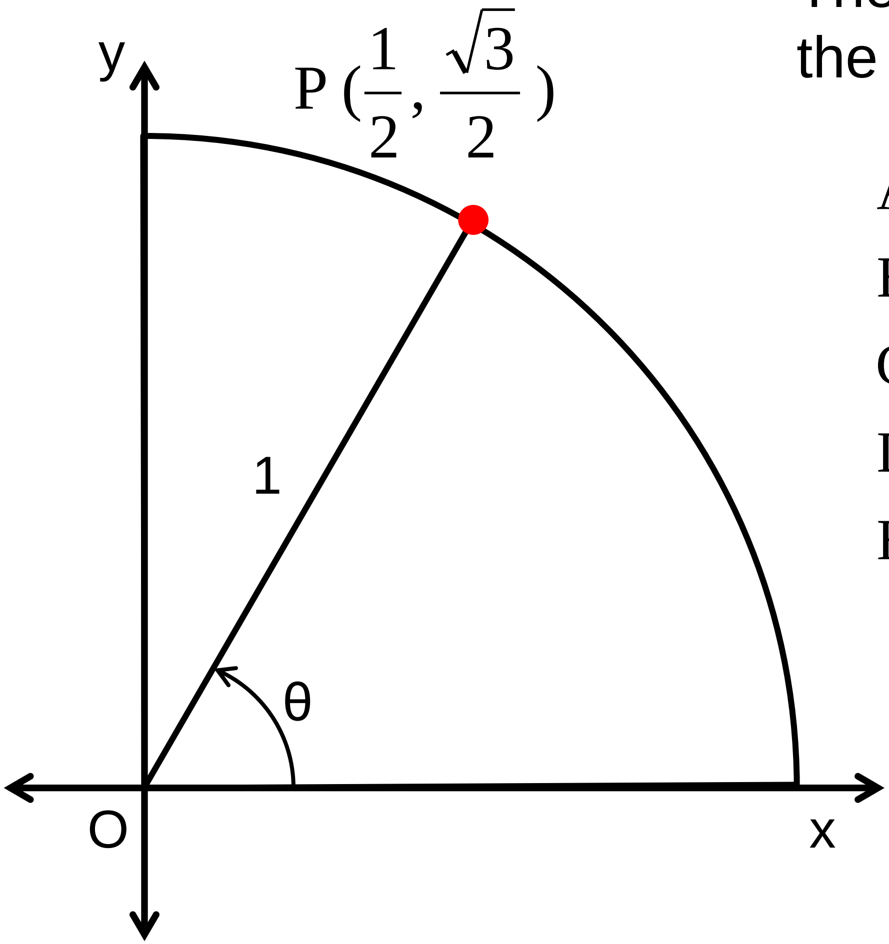
$$P(\cos 30°,\ \sin 30°) = P\left(\frac{\sqrt{3}}{2},\ \frac{1}{2}\right)$$

(From question 1) (From question 2)

$$\sin 30° = \frac{1}{2}, \quad \cos 30° = \frac{\sqrt{3}}{2}$$

You should now also be able to find the exact values of sin(60°) and cos(60°) using the 30-60-90 triangle and the unit circle. If not, review the previous questions.

The Unit Circle IV



The coordinates of point P are shown in the diagram. Determine the angle θ.

A. $\theta = 30°$

B. $\theta = 45°$

C. $\theta = 60°$

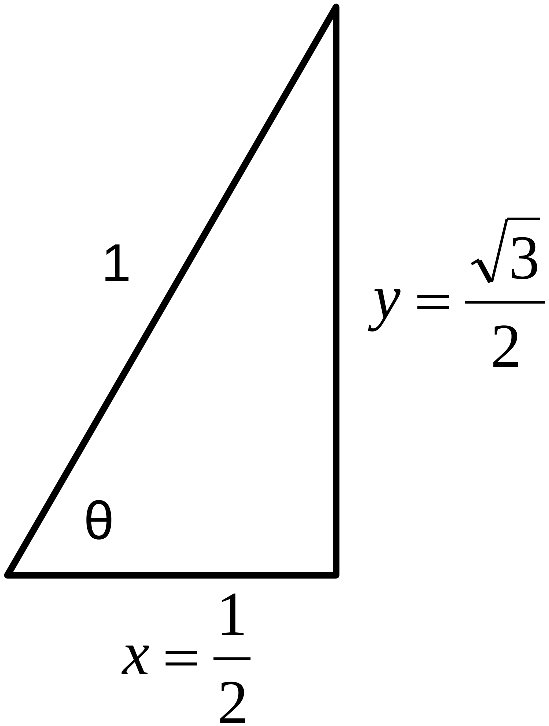
D. $\theta = 75°$

E. Cannot be done without a calculator

Solution

Answer: C

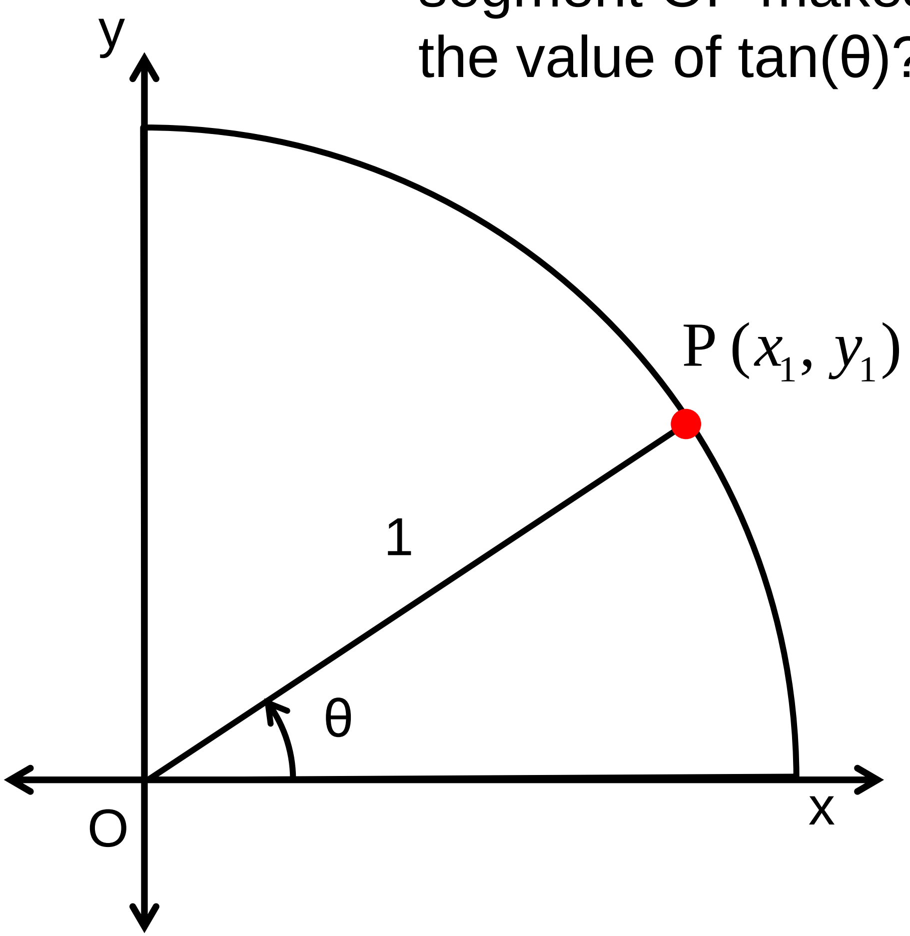
Justification: The coordinates of P are given, so we can draw the following triangle:



The ratio between the side lengths of the triangle are the same as a 30-60-90 triangle. This shows that θ = 60°.

The Unit Circle V

Consider an arbitrary point P on the unit circle. The line segment OP makes an angle θ with the x-axis. What is the value of tan(θ)?



A. $\tan(\theta) = \frac{1}{x_1}$

B. $\tan(\theta) = \frac{1}{y_1}$

C. $\tan(\theta) = \frac{x_1}{y_1}$

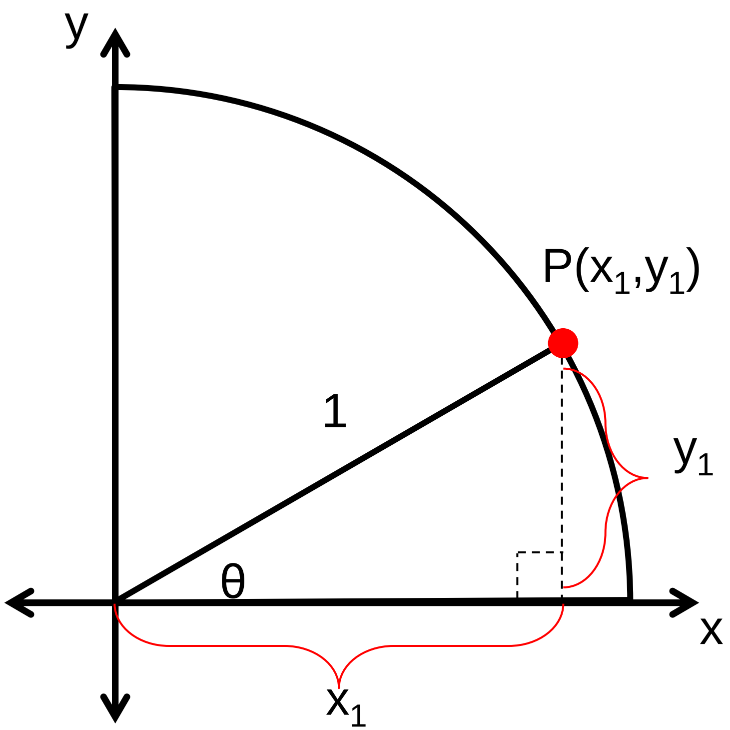
D. $\tan(\theta) = \frac{y_1}{x_1}$

E. Cannot be determined

Solution

Answer: D

Justification: Recall that: $x_1 = \cos(\theta), \quad y_1 = \sin(\theta)$



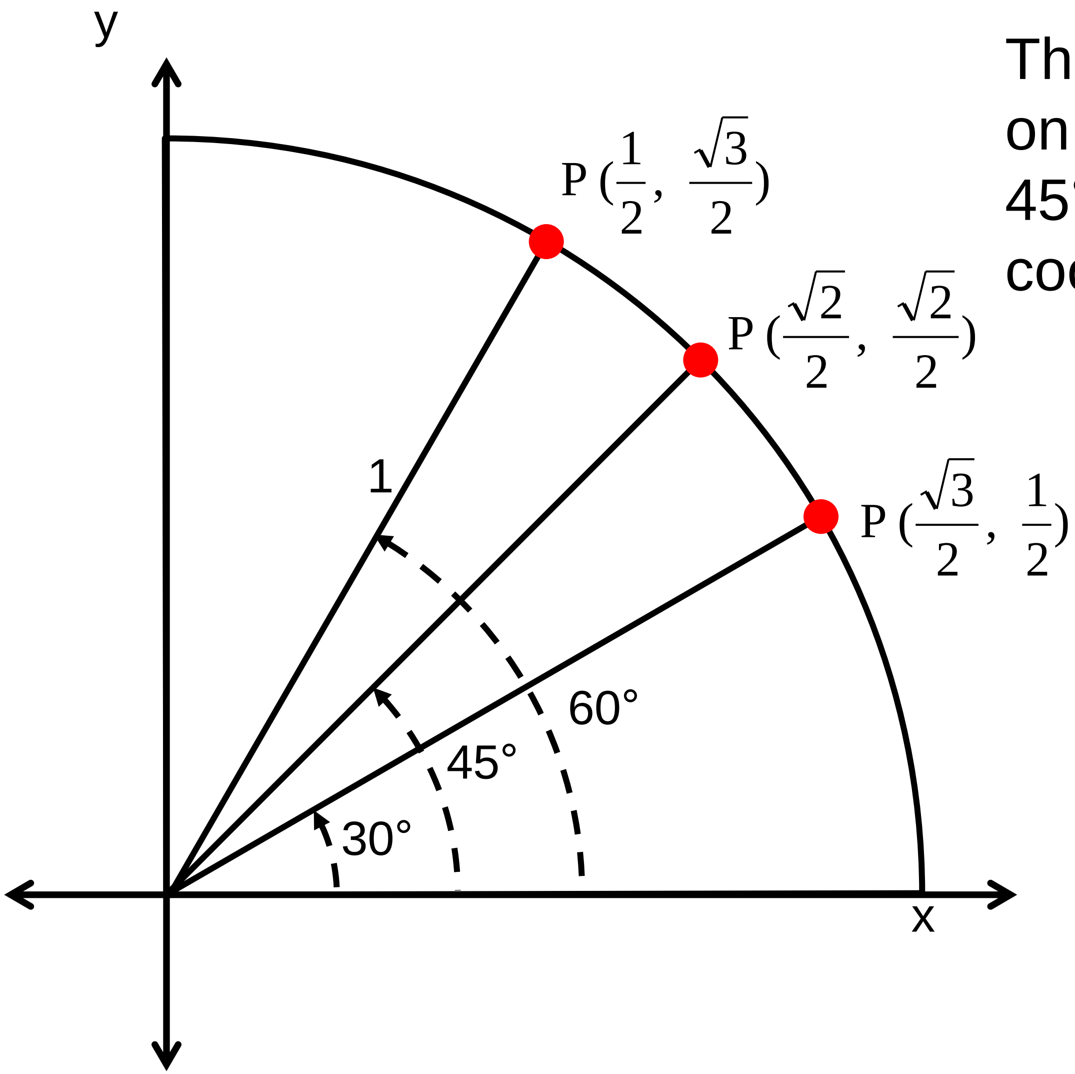
Since the tangent ratio of a right triangle can be found by dividing the opposite side by the adjacent side, the diagram shows:

$$\tan(\theta) = \frac{y_1}{x_1}$$

Using the formulas for x_1 and y_1 shown above, we can also define tangent as:

$$\tan(\theta) = \frac{\sin(\theta)}{\cos(\theta)}$$

Summary



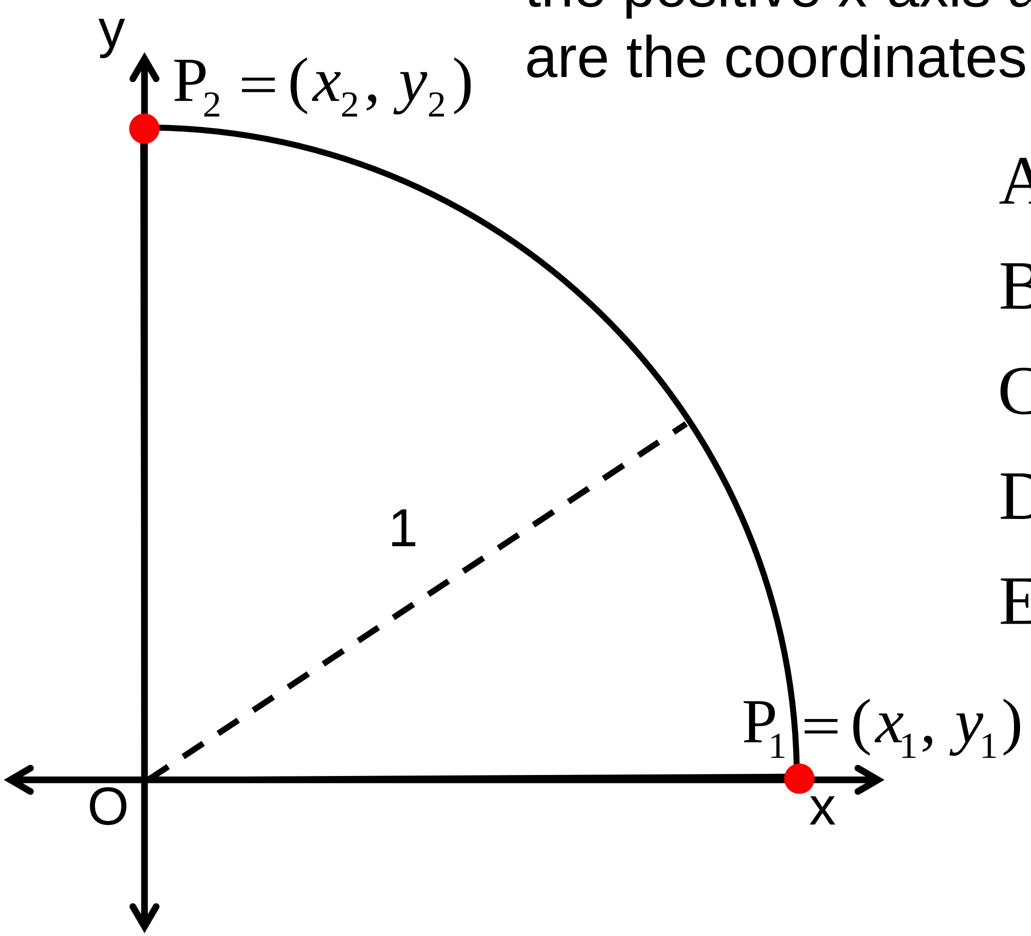
The diagram shows the points on the unit circle with θ = 30°, 45°, and 60°, as well as their coordinate values.

Summary

The following table summarizes the results from the previous questions.

	$\theta = 30°$	$\theta = 45°$	$\theta = 60°$
$\sin(\theta)$	$\frac{1}{2}$	$\frac{\sqrt{2}}{2}$	$\frac{\sqrt{3}}{2}$
$\cos(\theta)$	$\frac{\sqrt{3}}{2}$	$\frac{\sqrt{2}}{2}$	$\frac{1}{2}$
$\tan(\theta)$	$\frac{1}{\sqrt{3}}$	1	$\sqrt{3}$

The Unit Circle VI



Consider the points where the unit circle intersects the positive x-axis and the positive y-axis. What are the coordinates of P_1 and P_2?

A. $P_1 = (1, 0), \quad P_2 = (0, 1)$

B. $P_1 = (0, 1), \quad P_2 = (1, 0)$

C. $P_1 = (1, 1), \quad P_2 = (0, 0)$

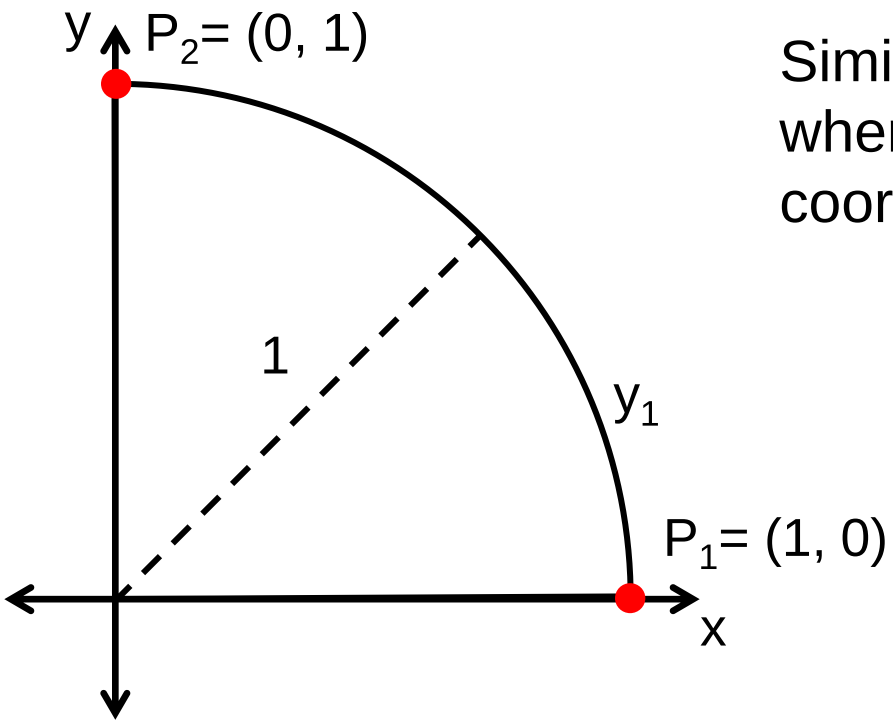
D. $P_1 = (0, 0), \quad P_2 = (1, 1)$

E. None of the above

Solution

Answer: A

Justification: Any point on the x-axis has a y-coordinate of 0. P_1 is on the x-axis as well as the unit circle which has radius 1, so it must have coordinates (1,0).

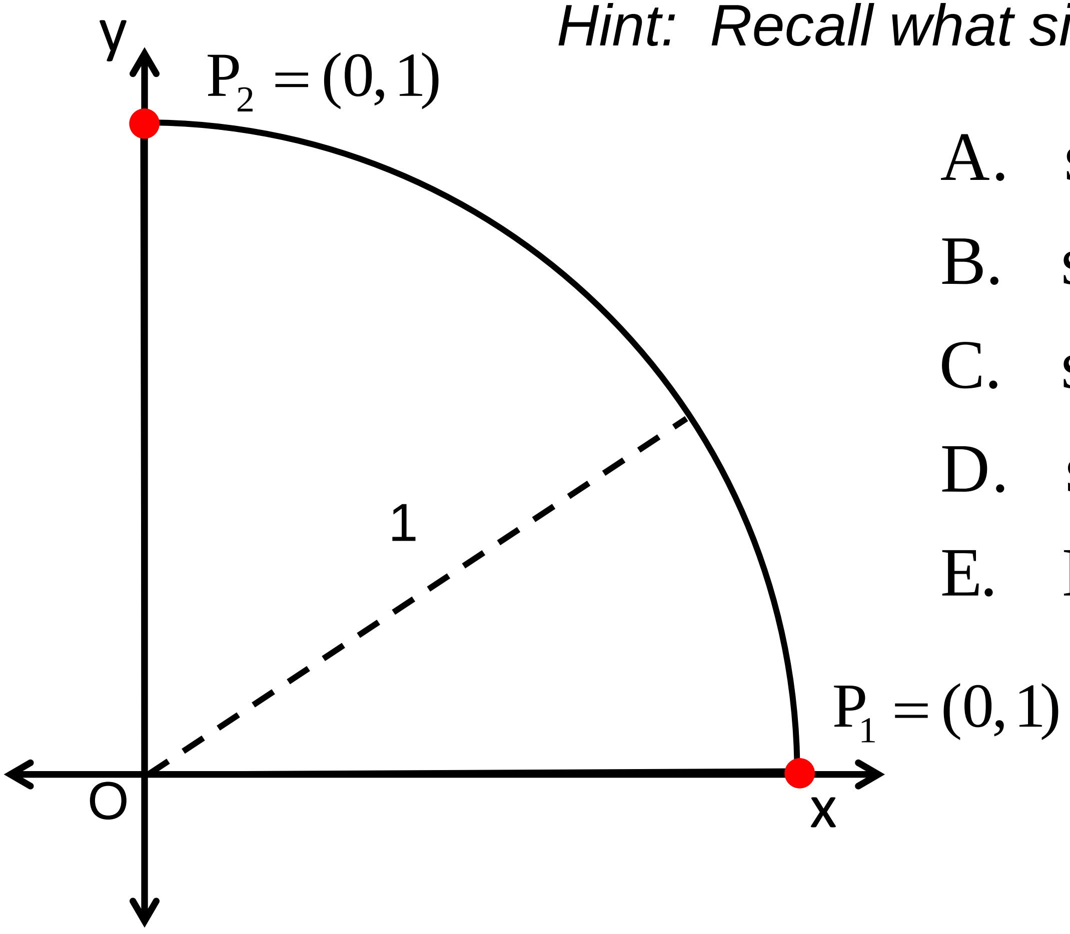


Similarly, any point is on the y-axis when the x-coordinate is 0. P_2 has coordinates (0, 1).

The Unit Circle VII

What are values of sin(0°) and sin(90°)?

Hint: Recall what sinθ represents on the graph



A. $\sin(0°) = 0, \quad \sin(90°) = 0$

B. $\sin(0°) = 1, \quad \sin(90°) = 0$

C. $\sin(0°) = 0, \quad \sin(90°) = 1$

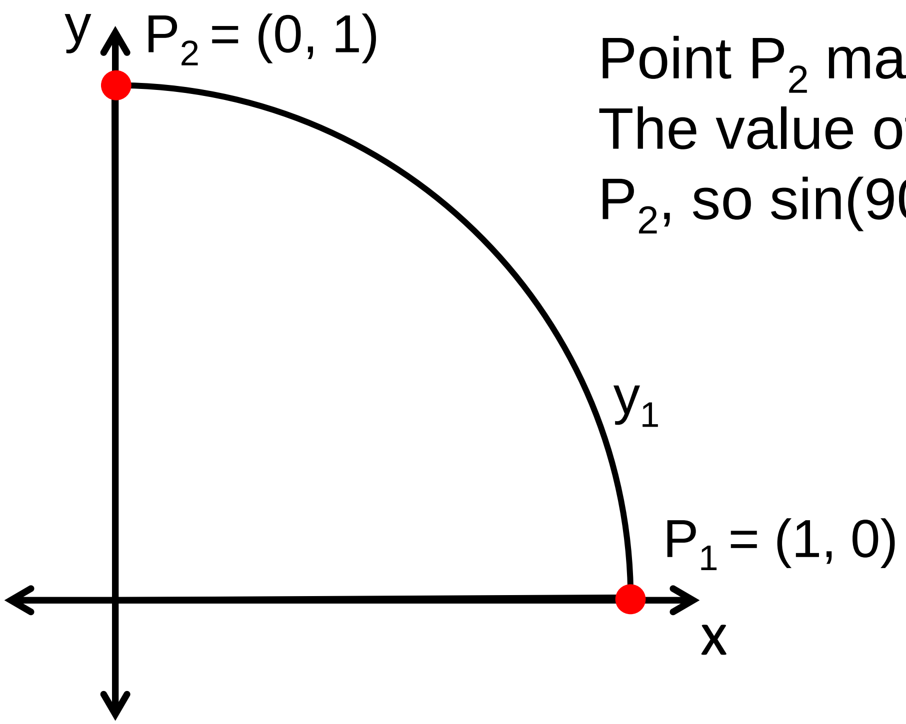
D. $\sin(0°) = 1, \quad \sin(90°) = 1$

E. None of the above

Solution

Answer: C

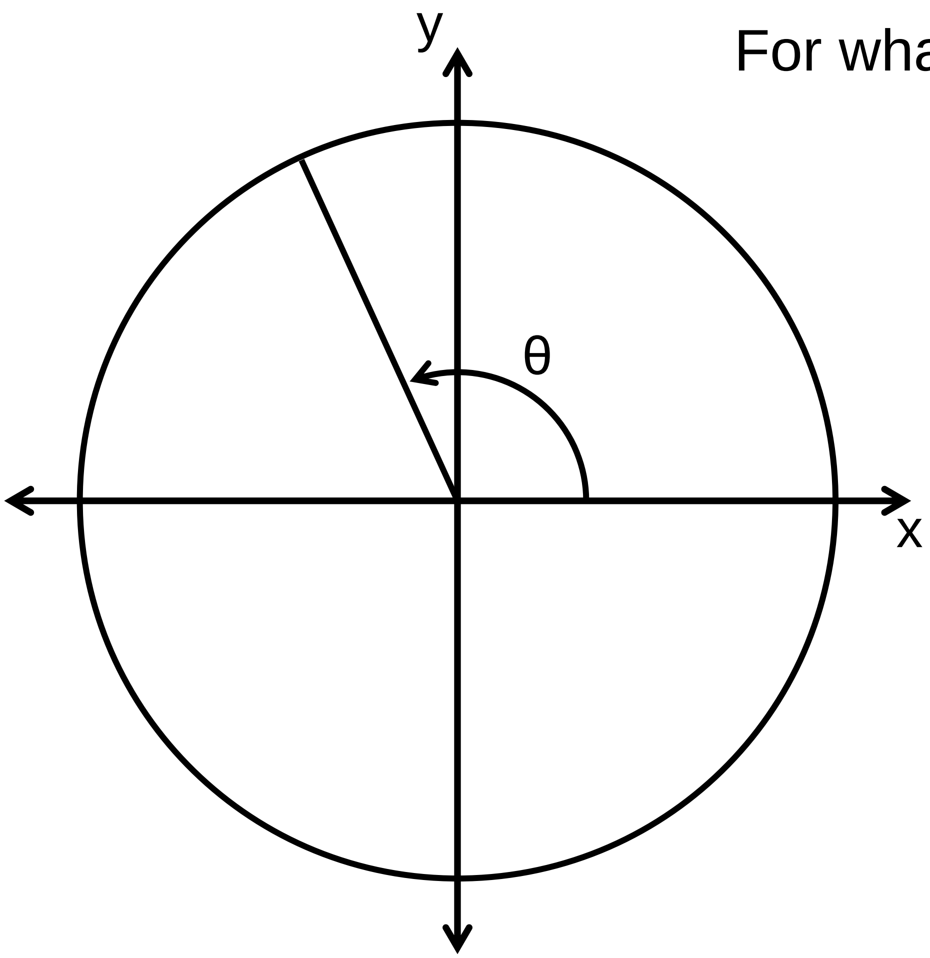
Justification: Point P_1 makes a 0° angle with the x-axis. The value of sin(0°) is the y-coordinate of P_1, so sin(0°) = 0.



Point P_2 makes a 90° angle with the x-axis. The value of sin(90°) is the y-coordinate of P_2, so sin(90°) = 1.

Try finding cos(0°) and cos(90°).

The Unit Circle VIII



For what values of θ is sin(θ) positive?

A. $0° < \theta < 90°$

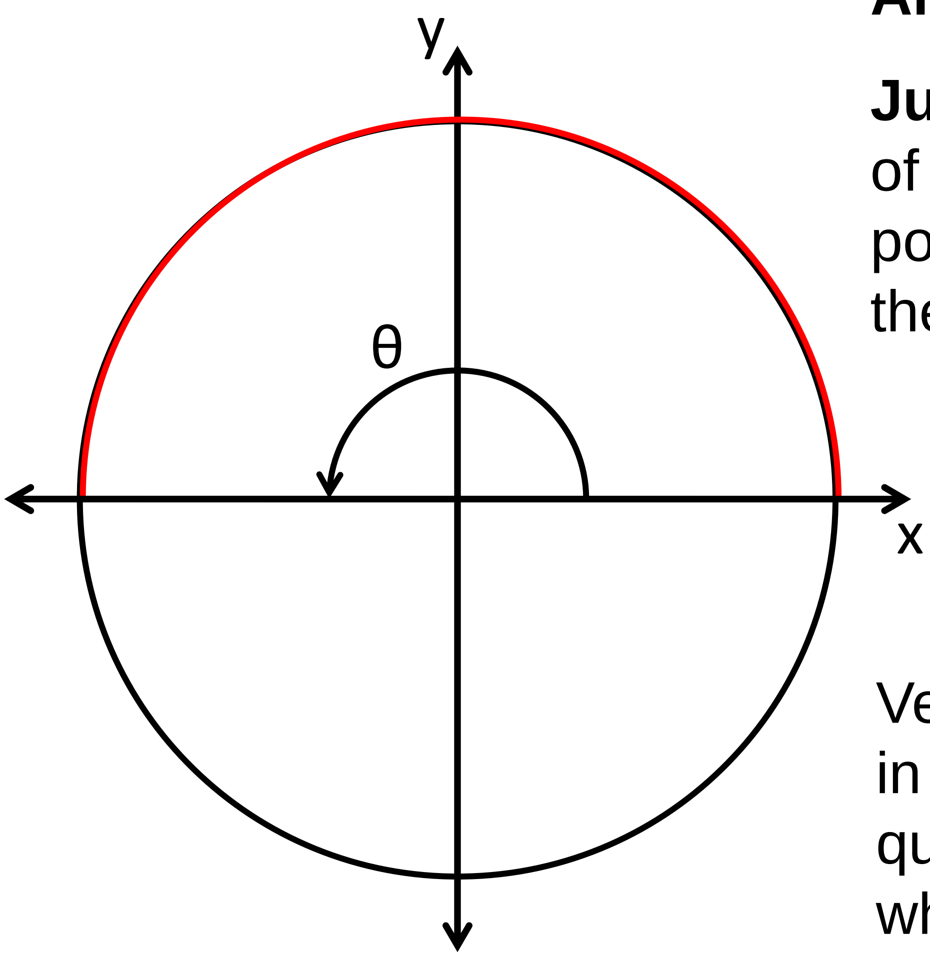
B. $0° < \theta < 180°$

C. $-90° < \theta < 90°$

D. $0° < \theta < 360°$

E. None of the above

Solution

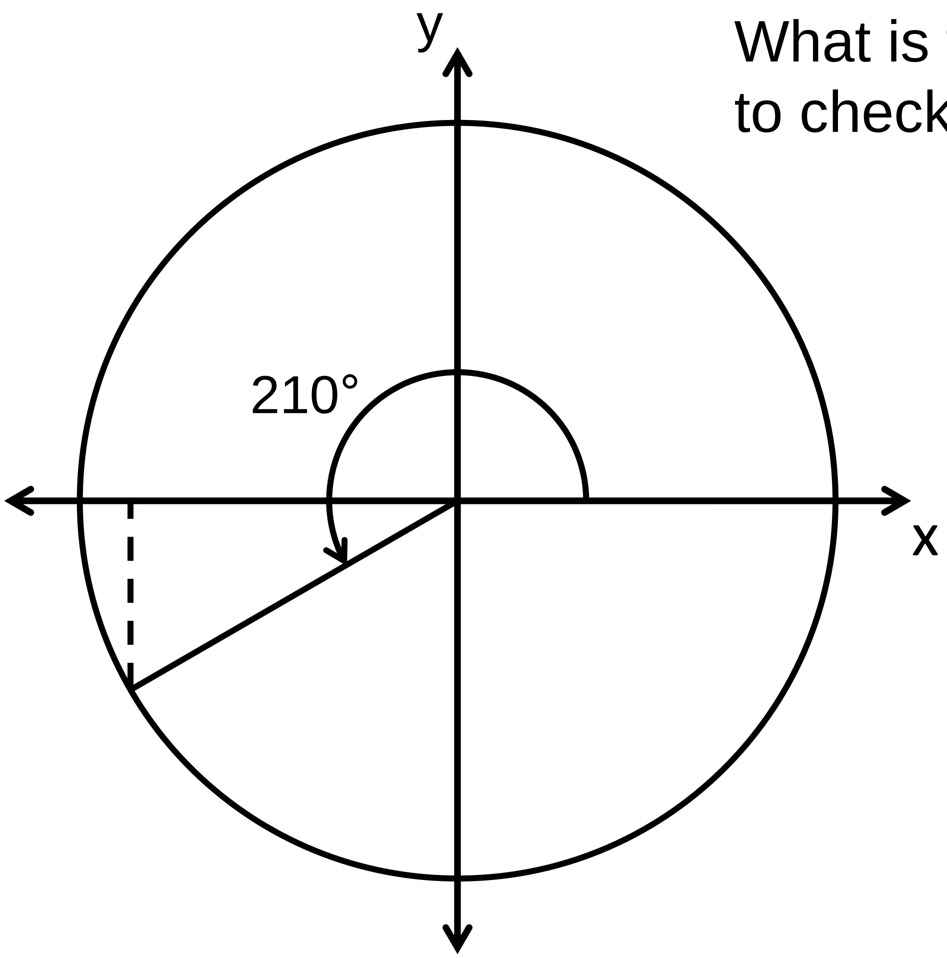


Answer: B

Justification: When the y coordinate of the points on the unit circle are positive (the points highlighted red) the value of sin(θ) is positive.

Verify that cos(θ) is positive when θ is in quadrant I (0°<θ<90°) and quadrant IV (270°< θ<360°), which is when the x-coordinate of the points is positive.

The Unit Circle IX



What is the value of sin(210°)? Remember to check if sine is positive or negative.

A. $\frac{1}{2}$

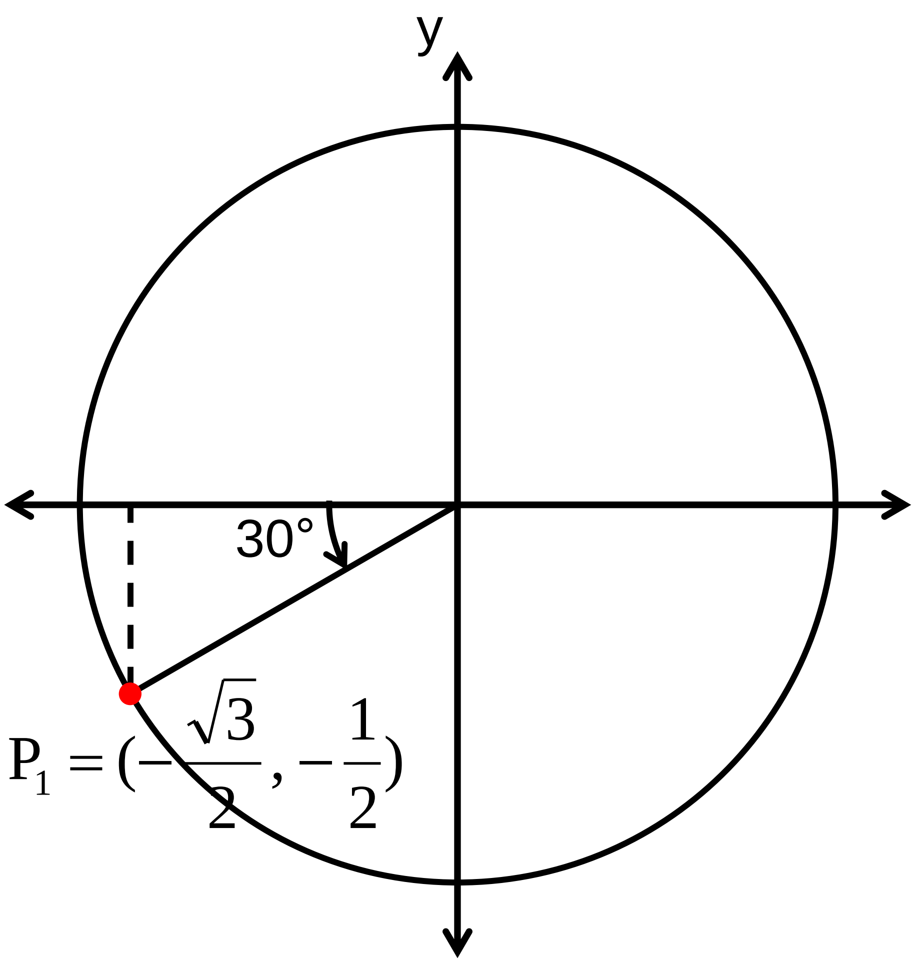
B. $-\frac{1}{2}$

C. $\frac{\sqrt{3}}{2}$

D. $-\frac{\sqrt{3}}{2}$

E. None of the above

Solution



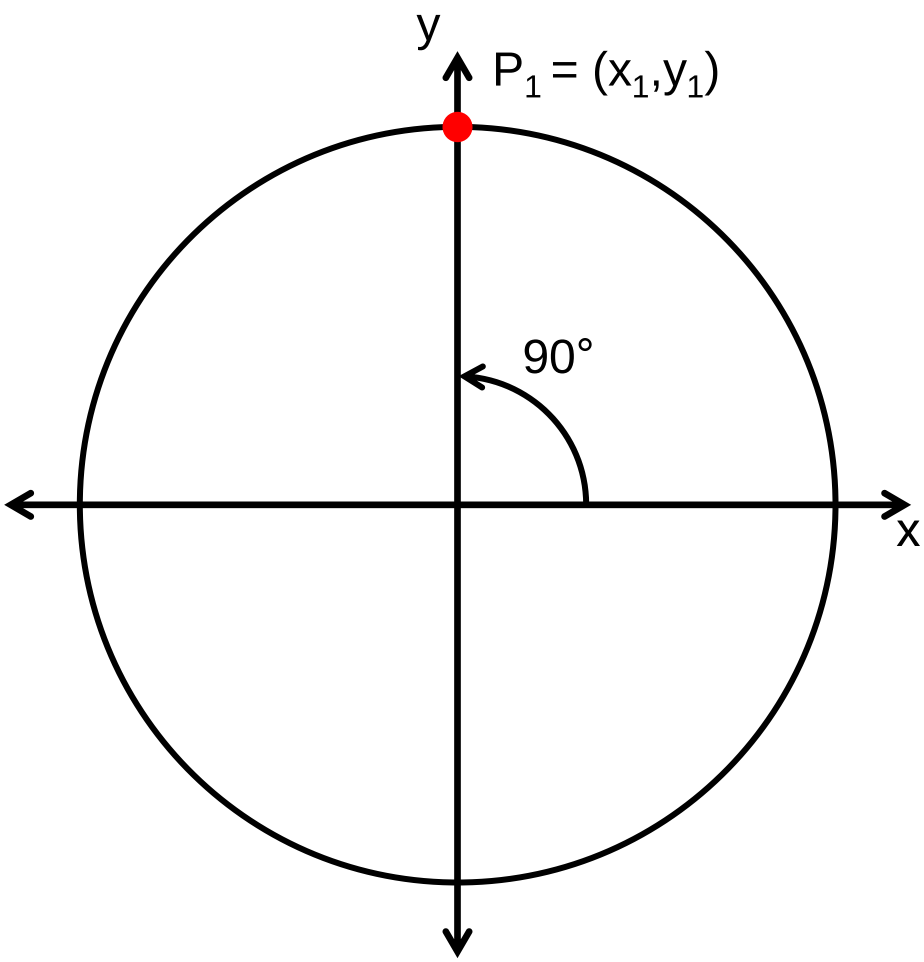
Answer: B

Justification: The point P_1 is below the x-axis so its y-coordinate is negative, which means sin(θ) is negative. The angle between P_1 and the x-axis is 210° – 180° = 30°.

$$\sin(210°) = -\sin(30°)$$

$$= -\frac{1}{2}$$

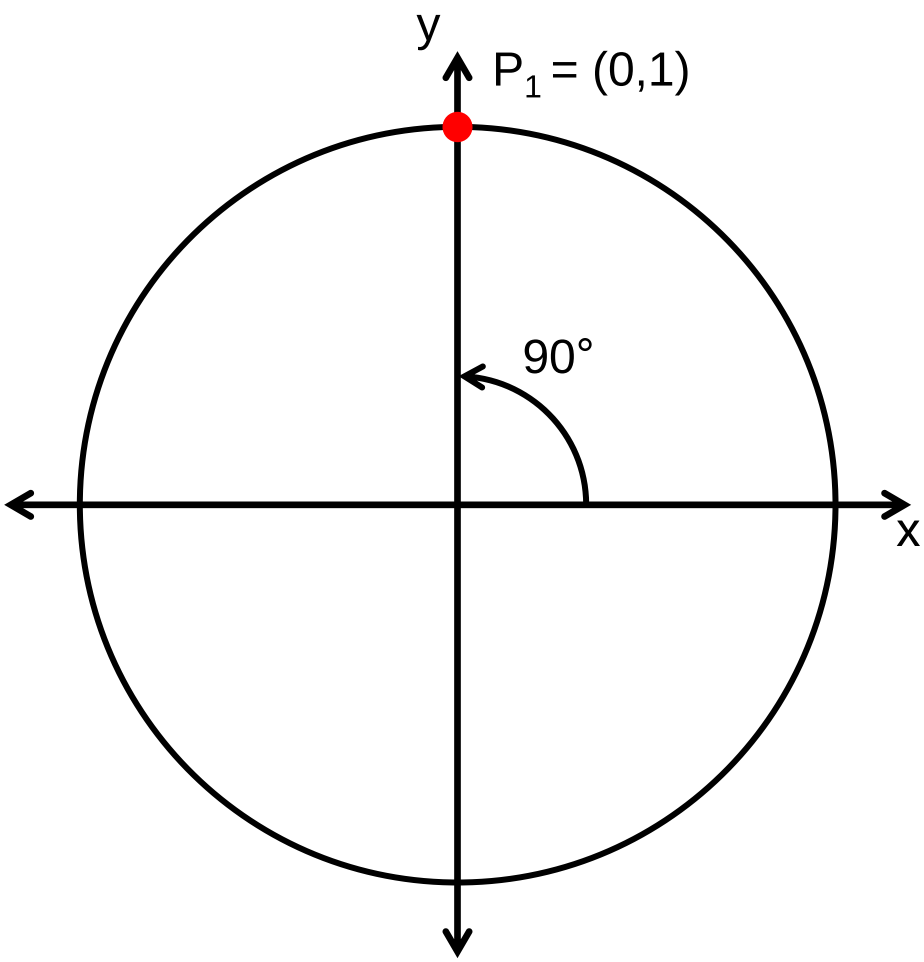
The Unit Circle X



What is the value of tan(90°)?

A. 0
B. 1
C. $\sqrt{3}$
D. $\frac{1}{\sqrt{3}}$
E. None of the above

Solution



Answer: E

Justification: Recall that the tangent of an angle is defined as:

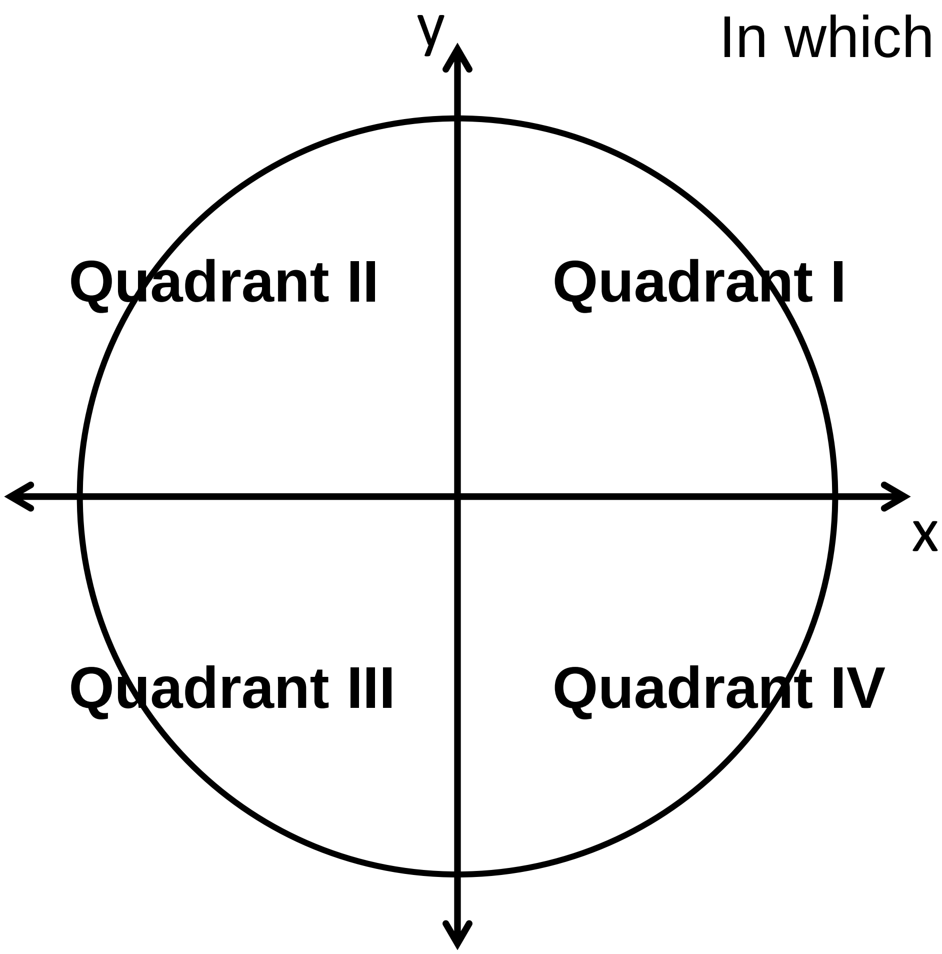
$$\tan\theta = \frac{\sin\theta}{\cos\theta}$$

When θ=90°,

$$\tan 90° = \frac{\sin 90°}{\cos 90°} = \frac{y_1}{x_1} = \frac{1}{0}$$

Since we cannot divide by zero, tan90° is undefined.

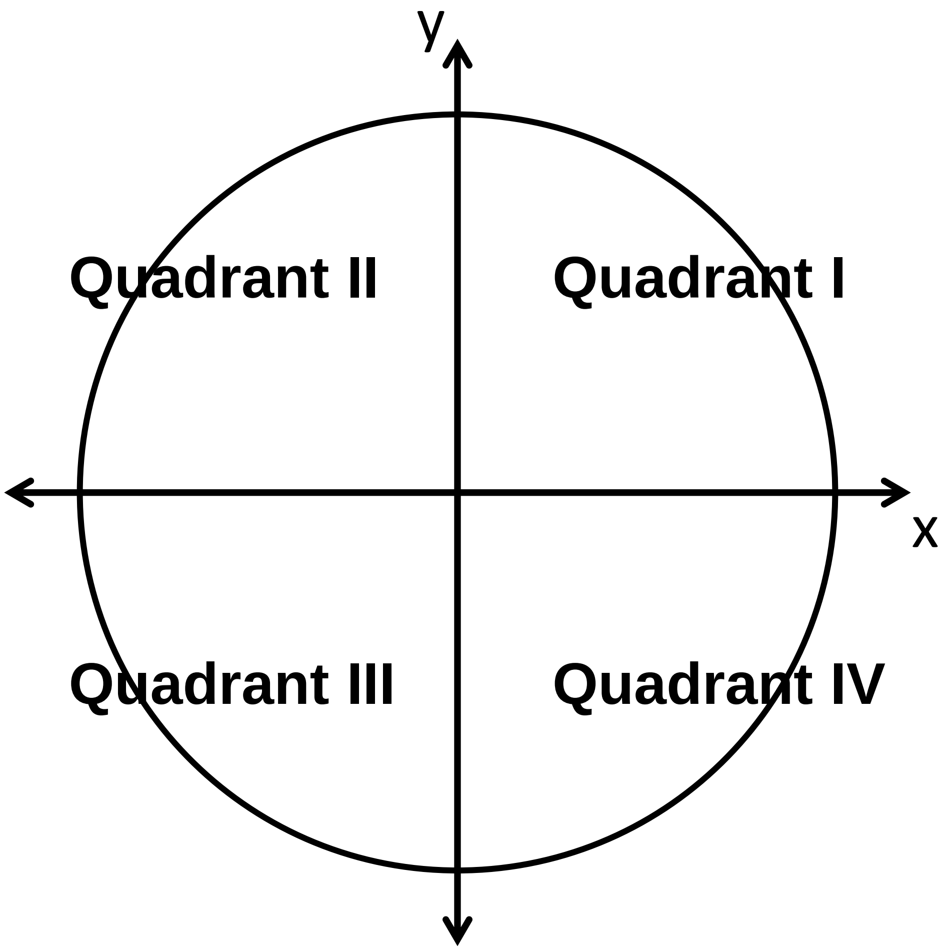
The Unit Circle XI



In which quadrants will tan(θ) be negative?

A. II
B. III
C. IV
D. I and III
E. II and IV

Solution



Answer: B

Justification: In order for tan(θ) to be negative, sin(θ) and cos(θ) must have opposite signs. In the 2nd quadrant, sine is positive while cosine is negative. In the 4th quadrant, sine is negative while cosine is positive.

Therefore, tan(θ) is negative in the 2nd and 4th quadrants.

Summary

The following table summarizes the results from the previous questions.

	θ = 0	θ = 90°	θ = 180°	θ = 270°
sin(θ)	0	1	0	-1
cos(θ)	1	0	-1	0
tan(θ)	0	undefined	0	undefined

Summary

Quadrant II

$\sin(\theta) > 0$

$\cos(\theta) < 0$

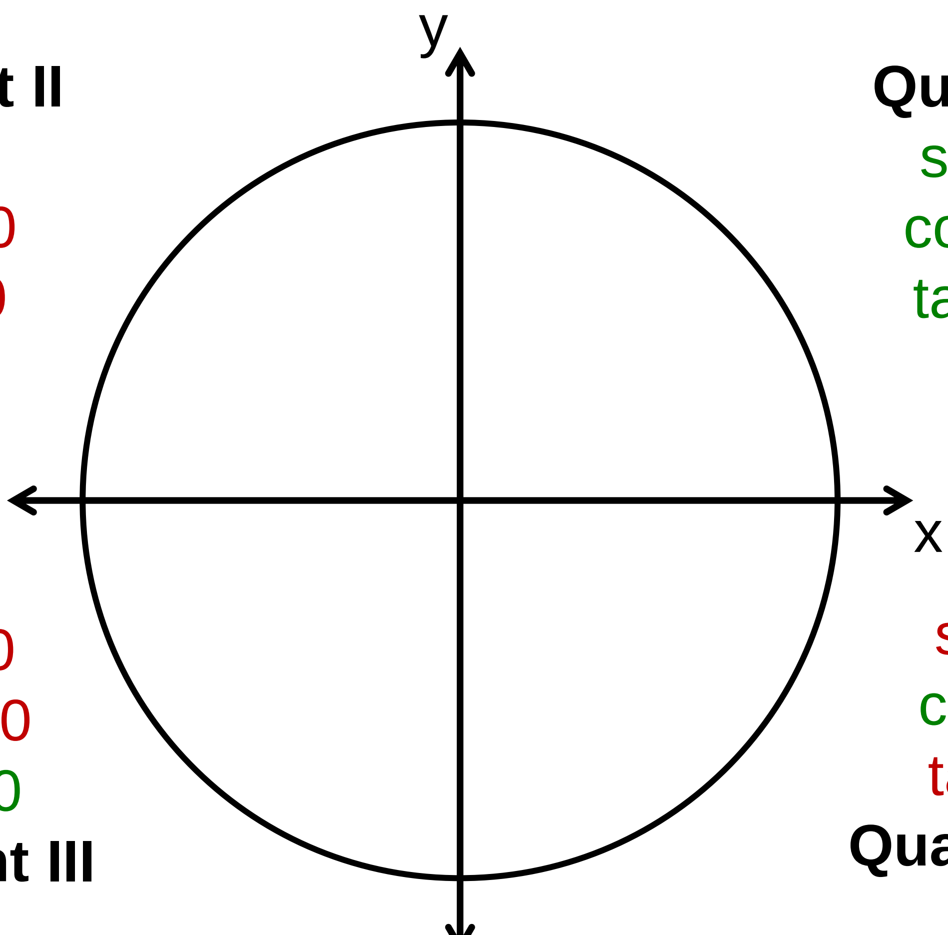
$\tan(\theta) < 0$

Quadrant I

$\sin(\theta) > 0$

$\cos(\theta) > 0$

$\tan(\theta) > 0$



$\sin(\theta) < 0$

$\cos(\theta) < 0$

$\tan(\theta) > 0$

Quadrant III

$\sin(\theta) < 0$

$\cos(\theta) > 0$

$\tan(\theta) < 0$

Quadrant IV