a place of mind

FACULTY OF EDUCATION

Department of
Curriculum and Pedagogy

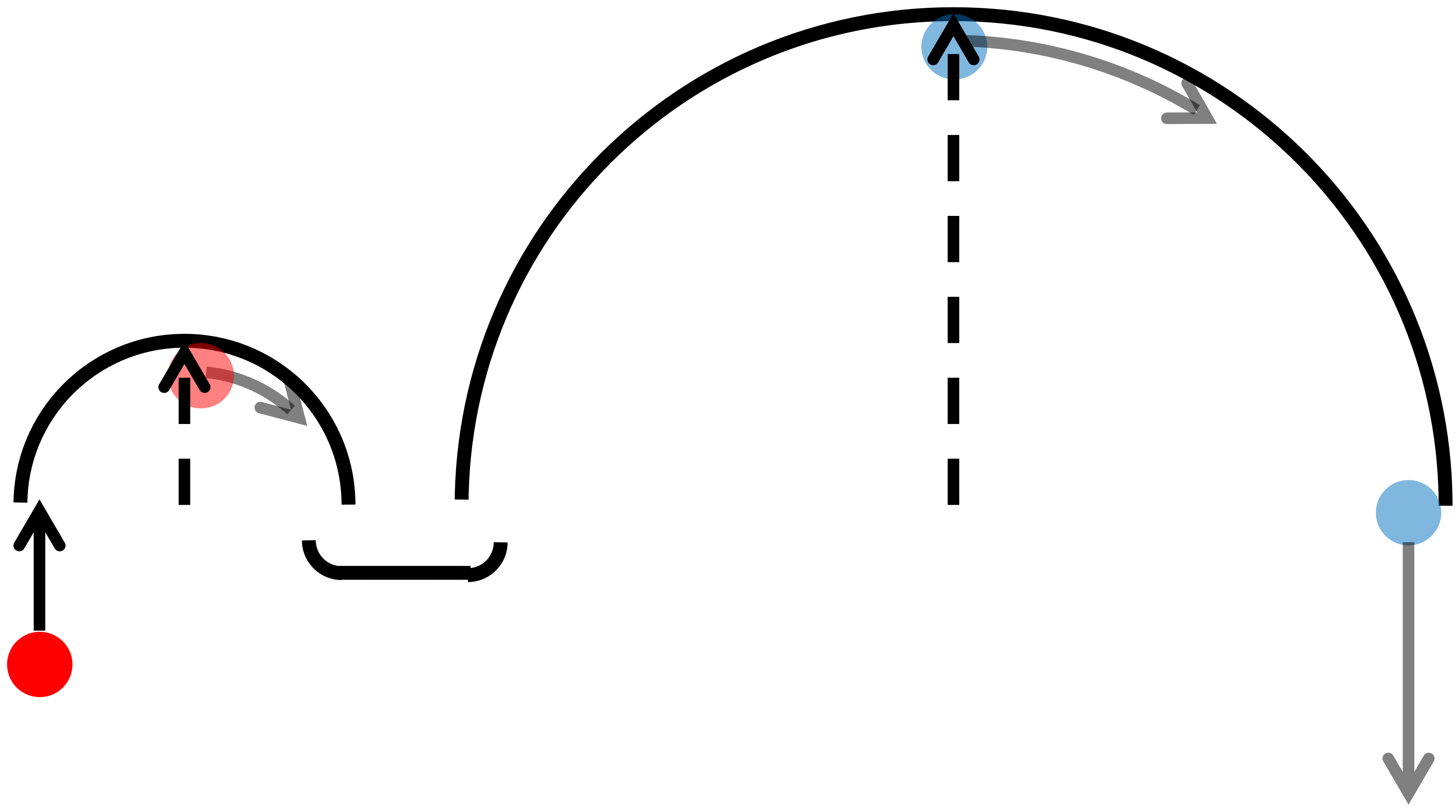
Physics

Circular Motion: Energy and Momentum Conservation

Science and Mathematics
Education Research Group

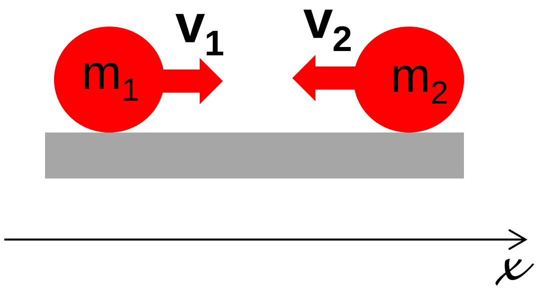
Supported by UBC Teaching and Learning Enhancement Fund 2012-2013

Semicircles IV



Elastic Collisions

This is a challenging set for the students who are interested in physics and like challenges. However, it is also a very beautiful set which will help students build their intuition. In order to be able to solve questions related to circular motion, the students have to know how to do collision problems. Therefore, we review elastic collisions in the first part of the set and then move to discuss circular motion



Perfectly Elastic Collisions I

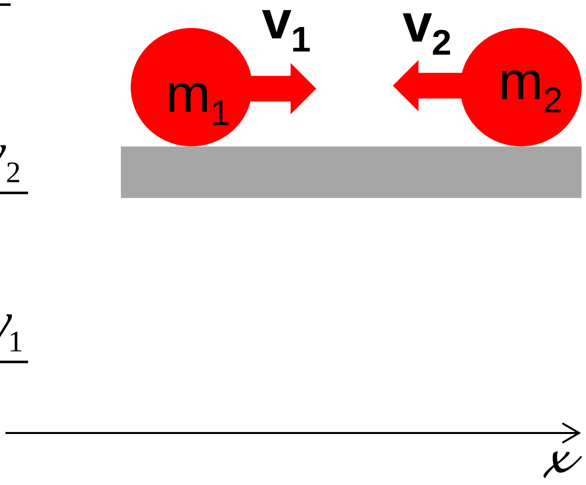
Two balls with masses of m_1 and m_2 are moving towards each other with speeds of v_1 and v_2 respectively. If the collision between the balls is a perfectly elastic collision, what will the speeds of the balls be after the collision?

$$A.\begin{cases} u_1 = \dfrac{v_1(m_1 - m_2) + 2m_2v_2}{m_1 + m_2} \\ u_2 = \dfrac{v_2(m_2 - m_1) + 2m_1v_1}{m_1 + m_2} \end{cases}$$

$$C.\begin{cases} u_1 = \dfrac{v_1(m_1 - m_2) + 2m_2v_2}{(m_1 + m_2)^2} \\ u_2 = \dfrac{v_2(m_2 - m_1) + 2m_1v_1}{(m_1 + m_2)^2} \end{cases}$$

$$B.\begin{cases} u_1 = \dfrac{v_1(m_1 - m_2) + 2m_2v_2^2}{m_1 + m_2} \\ u_2 = \dfrac{v_2(m_2 - m_1) + 2m_1v_1^2}{m_1 + m_2} \end{cases}$$

$$D.\begin{cases} u_1 = \dfrac{v_1(m_1 - m_2)^2 + 2m_2v_2}{m_1 + m_2} \\ u_2 = \dfrac{v_2(m_2 - m_1)^2 + 2m_1v_1}{m_1 + m_2} \end{cases}$$



Solution

Answer: A

Justification: We can find correct answer by applying the laws of energy and momentum conservation as shown below. However, since it is a multiple-choice question, we can also attempt to eliminate the wrong answers. Answers B-D all have incorrect units. You can see that one of the terms in them is squared thus making the units in the numerator or in the denominator incorrect or not matching.

$$\begin{cases} m_1 v_1 + m_2 v_2 = m_1 u_1 + m_2 v_2 \\ \dfrac{m_1 v_1^2}{2} + \dfrac{m_2 v_2^2}{2} = \dfrac{m_1 u_1^2}{2} + \dfrac{m_2 u_2^2}{2} \end{cases} \Rightarrow \begin{cases} u_1 = \dfrac{v_1 \left(m_1 - m_2 \right) + 2 m_2 v_2}{m_1 + m_2} \\ u_2 = \dfrac{v_2 \left(m_2 - m_1 \right) + 2 m_1 v_1}{m_1 + m_2} \end{cases}$$

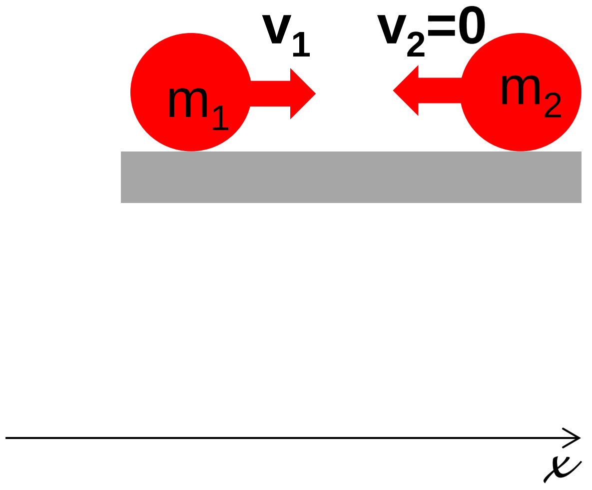
Perfectly Elastic Collisions II

A ball with mass m_1 is moving towards another ball with mass m_2 . The second ball is initially at rest ($v_2 = 0$). If the collision between the balls is a perfectly elastic collision, what will the speeds of the balls be after the collision?

$$A.\begin{cases} u_1 = \dfrac{2m_2 v_2}{m_1 + m_2} \\ u_2 = \dfrac{v_2(m_2 - m_1)}{m_1 + m_2} \end{cases} \qquad C.\begin{cases} u_1 = \dfrac{m_2}{m_1} v_2 \\ u_2 = \dfrac{m_2}{m_1} v_1 \end{cases}$$

$$B.\begin{cases} u_1 = \dfrac{v_1(m_1 - m_2)}{m_1 + m_2} \\ u_2 = \dfrac{2m_1 v_1}{m_1 + m_2} \end{cases} \qquad D.\begin{cases} u_1 = \dfrac{v_1(m_1 + m_2)}{m_1 - m_2} \\ u_2 = \dfrac{2m_1 v_1}{m_1 - m_2} \end{cases}$$

E. None of the above



Solution

Answer: B

Justification: We can find correct answer by using the collision equation from the previous question and assuming that v_2=0 (initial velocity of the second ball is zero):

$$\text{Since } v_2 = 0:$$

$$\begin{cases} u_1 = \dfrac{v_1(m_1 - m_2) + 2m_2 v_2}{m_1 + m_2} = \dfrac{v_1(m_1 - m_2)}{m_1 + m_2} \\ u_2 = \dfrac{v_2(m_2 - m_1) + 2m_1 v_1}{m_1 + m_2} = \dfrac{2m_1 v_1}{m_1 + m_2} \end{cases}$$

Notice, the answers are not symmetrical (which makes sense as we had asymmetrical initial conditions. You can check that the correct answer also has correct units. In both cases, the denominator represents the mass of the system. The numerator of the first equation represents the difference of masses, we will explore its meaning in the next question.

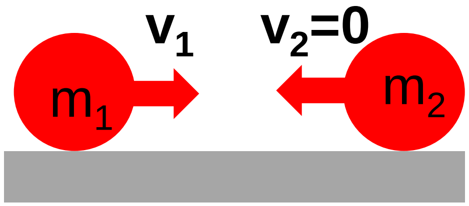
Perfectly Elastic Collisions III

A ball with mass m_1 is moving towards another ball with mass m_2 . The second ball is initially at rest ($v_2 = 0$). If the collision between the balls is a perfectly elastic collision and $m_1 = m_2$, what will the speeds of the balls be after the collision?

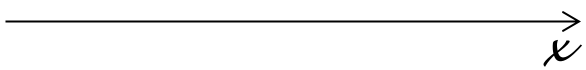
$$A.\begin{cases} u_1 = 0 \\ u_2 = -v_1 \end{cases} \qquad C.\begin{cases} u_1 = 2v_1 \\ u_2 = v_1 \end{cases}$$

$$B.\begin{cases} u_1 = -2v_1 \\ u_2 = -v_1 \end{cases} \qquad \text{D.}\begin{cases} u_1 = v_1 \\ u_2 = 0 \end{cases}$$

$$\text{E.}\begin{cases} u_1 = 0 \\ u_2 = v_1 \end{cases}$$



$$m_1 = m_2 = m$$



Solution

Answer: B

Justification: We can find correct answer by using the collision equation from the previous question and assuming that not only v_2=0 (initial velocity of the second ball is zero) but also that

Assuming that initial velocity of the second ball is zero: $v_2 = 0$ and $m_1 = m_2 = m$

$$\begin{cases} u_1 = \dfrac{v_1\left(m_1 - m_2\right) + 2m_2 v_2}{m_1 + m_2} = 0 \\ u_2 = \dfrac{v_2\left(m_1 - m_2\right) + 2m_1 v_1}{m_1 + m_2} = \dfrac{2mv_1}{2m} = v_1 \end{cases}$$

$$m_1 = m_2 \equiv \mathrm{m}$$

This tells us that the ball that was moving initially will stop and the ball that was originally at rest will start moving with the initial speed of the first ball. This phenomenon is used in a famous demonstration called Newton's cradle: https://www.youtube.com/watch?v=0LnbyjOyEQ8

You can see that none of the other answers make sense!

Perfectly Elastic Collisions IV

A ball with mass m_1 is moving towards another ball with mass m_2 . The second ball is initially at rest ($v_2 = 0$). If the collision between the balls is a perfectly elastic collision and $m_2 >> m_1$, what will the speeds of the balls be after the collision?

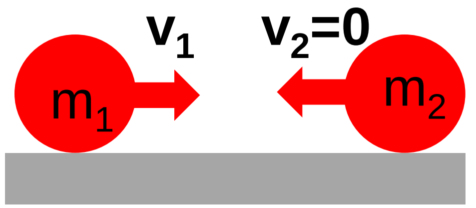
A. $\begin{cases} u_1 = -v_1 \\ u_2 = 2v_1 \end{cases}$

B. $\begin{cases} u_1 = -2v_1 \\ u_2 = -v_1 \end{cases}$

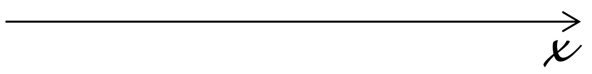
C. $\begin{cases} u_1 = 2v_1 \\ u_2 = v_1 \end{cases}$

D. $\begin{cases} u_1 = -v_1 \\ u_2 = 0 \end{cases}$

E. $\begin{cases} u_1 = v_1 \\ u_2 = 2v_1 \end{cases}$



$$m_2 >> m_1$$



Solution

Answer: D

Justification: We can find correct answer by using the collision equation from the previous question and assuming that not only v_2=0 (initial velocity of the second ball is zero) but also that

$$m_2 >> m_1 \text{ or } \frac{m_1}{m_2} << 1$$

Since $v_2 = 0$ and $\frac{m_1}{m_2} << 1$:

$$\begin{cases} u_1 = \dfrac{v_1(m_1 - m_2) + 2m_2 v_2}{m_1 + m_2} = \dfrac{v_1\left(\dfrac{m_1}{m_2} - 1\right)}{1 + \dfrac{m_1}{m_2}} \approx -v_1 \\ u_2 = \dfrac{v_2(m_2 - m_1) + 2m_1 v_1}{m_1 + m_2} = \dfrac{2m_1 v_1}{m_1 + m_2} = \dfrac{2\dfrac{m_1}{m_2} v_1}{1 + \dfrac{m_1}{m_2}} \approx 0 \end{cases}$$

This is a very interesting conclusion. While the heavy ball m_2 will practically stop, the light ball will bounce off it with the speed equal to its initial speed. This makes sense considering our earlier discussion.

You can see that none of the other answers make sense!

Perfectly Elastic Collisions V

A ball with mass m_1 is moving towards another ball with mass m_2. The second ball is initially at rest ($v_2 = 0$). If the collision between the balls is a perfectly elastic collision and $m_1 >> m_2$, what will the speeds of the balls be after the collision?

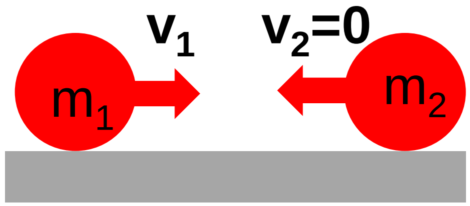
A. $\begin{cases} u_1 = -v_1 \\ u_2 = 2v_1 \end{cases}$

B. $\begin{cases} u_1 = -2v_1 \\ u_2 = -v_1 \end{cases}$

C. $\begin{cases} u_1 = 2v_1 \\ u_2 = v_1 \end{cases}$

D. $\begin{cases} u_1 = -v_1 \\ u_2 = -2v_1 \end{cases}$

E. $\begin{cases} u_1 = v_1 \\ u_2 = 2v_1 \end{cases}$



$$m_1 >> m_2$$

Solution

Answer: E

Justification: We can find correct answer by using the collision equation from the previous question and assuming that not only $v_2=0$ (initial velocity of the second ball is zero) but also that $m_1 >> m_2$ or $\frac{m_2}{m_1} << 1$

Since $v_2 = 0$ and $\frac{m_2}{m_1} << 1$:

$$\begin{cases} u_1 = \dfrac{v_1(m_1 - m_2) + 2m_2 v_2}{m_1 + m_2} = \dfrac{v_1\left(1 - \dfrac{m_2}{m_1}\right)}{1 + \dfrac{m_2}{m_1}} \approx v_1 \\ u_2 = \dfrac{v_2(m_2 - m_1) + 2m_1 v_1}{m_1 + m_2} = \dfrac{2m_1 v_1}{m_1 + m_2} = \dfrac{2v_1}{1 + \dfrac{m_2}{m_1}} \approx 2v_1 \end{cases}$$

While the heavy ball m_1 will continue moving almost unaffected, the light ball m_2 that was initially at rest will bounce off with the speed equal to twice the speed of the heavy ball $2v_1$. How come? This is easier to understand in the frame of reference of ball m_1. In that frame of reference, ball m_2 will be moving towards m_1 with the speed v_1 and as we discussed earlier it will bounce off with the speed v_1 relatively to ball m_1. However, since ball m_1 is moving relatively to the ground with the velocity v_1, the velocity of ball m_2 relatively to the ground will be $2v_1$.

Perfectly Elastic Collisions VI

Balls m_1 and m_2 are moving towards each other with equal speeds v relatively to the ground. If the collision between the balls is a perfectly elastic collision and $m_1 = m_2$, what will the speeds of the balls be after the collision?

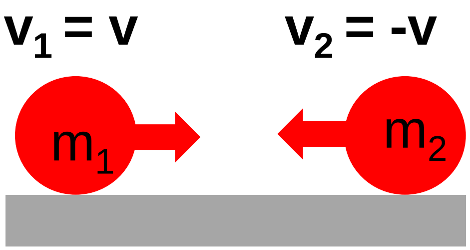
A. $\begin{cases} u_1 = -v \\ u_2 = v \end{cases}$

B. $\begin{cases} u_1 = v \\ u_2 = v \end{cases}$

C. $\begin{cases} u_1 = 2v \\ u_2 = -2v \end{cases}$

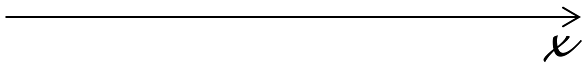
D. $\begin{cases} u_1 = -2v \\ u_2 = 2v \end{cases}$

E. $\begin{cases} u_1 = -v \\ u_2 = -v \end{cases}$



$$m_1 = m_2 = m$$

$$v_1 = -v_2$$



Solution

Answer: A

Justification: We can find correct answer by using the collision equation from the previous questions and assuming that not the velocities of the balls are opposite and their masses are equal:

$$m_1 = m_2 \equiv \mathrm{m}$$

$$v_1 = -v_2 \equiv v$$

$$\mathrm{v}_1 = -v_2 = v \text{ and } m_1 = m_2 = m$$

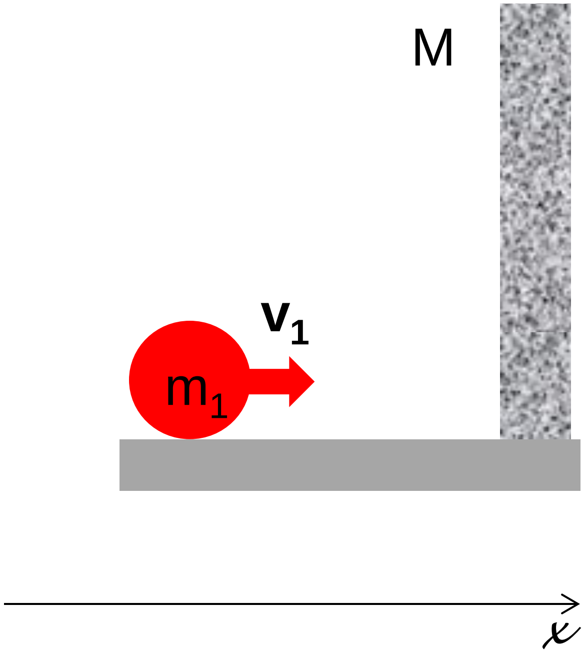
$$\begin{cases} u_1 = \dfrac{v_1(m_1 - m_2) + 2m_2v_2}{m_1 + m_2} = \dfrac{v(m - m) - 2mv}{2m} = -v \\ u_2 = \dfrac{v_2(m_2 - m_1) + 2m_1v_1}{m_1 + m_2} = \dfrac{v(m - m) + 2mv}{2m} = v \end{cases}$$

It makes sense that the balls will bounce off each other and will move in opposite directions with the same speed as they had before the collision.

You can see that none of the other answers make sense!

Perfectly Elastic Collisions VII

A ball with mass m_1 is moving towards a very big wall. If the collision between the ball and the wall is a perfectly elastic collision, what will be the result of the collision and what will happen to the wall? FIND THE **WRONG** STATEMENT:



A. The ball will bounce back with the speed v_1
B. The ball will bounce back with the speed $2v_1$
C. The wall will bounce back with the speed of v_1
D. The ball and the wall is NOT a closed system, so the momentum of the system will not be conserved
E. The wall will be compressed and it will exert a force on the ball that according to Newton's third law will be: $\vec{F}_{\text{ball on wall}} = -\vec{F}_{\text{wall on ball}}$

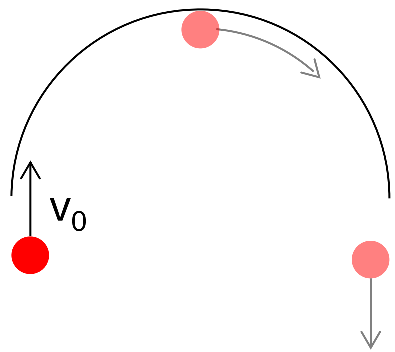
Solution

Answer: B

Justification: The only incorrect statement is B. The wall is connected to the ground, so the ball and wall is NOT a closed system. While the ball will not move, it will exert a force on the ball that will make the ball bounce back with the same speed it came with. Notice, the law of momentum conservation only works for two objects that interact with each other and are unaffected by other objects. In this case, the wall is affected by the ground. If the “wall” was on wheels, or was able to move back, then the wall would have moved with the speed of $2v_1$ and the ball would have bounced back with the speed of $2v_1$.

Part II: Circular Motion

The questions in this sub-set combine concepts of circular motion, energy and momentum conservation.



Semicircles VIII

A hollow and frictionless semicircle is placed vertically as shown below. A ball enters with an initial speed v_0 and exits out the other end. What is the final speed of the ball?

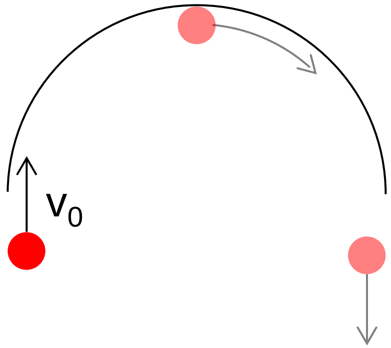
A. v_0

B. v_0/r

C. $v_0/2$

D. $v_0/3$

E. No idea



Solution

Answer: A

Justification: Since the semi-circle is frictionless, the mechanical energy of the ball has to be conserved.

The entrance and exit points of the semicircle are at the same height. This means the gravitational potential energy of the ball is equal at these two points. Therefore, the kinetic energies must also be the same, and the speed of the ball is the same upon entrance and exit of the semi circle.

Semicircles IX

A hollow and frictionless semicircle is placed vertically as shown below. A ball enters with an initial speed and exits out the other end. If the semicircle has a radius r, what is the minimum value of v_0 that will allow the ball to complete the path around the semicircle?

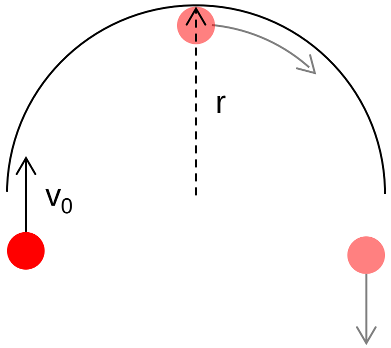
A. $\sqrt{gr}$

B. $\sqrt{2gr}$

C. $\sqrt{3gr}$

D. gr

E. $g\sqrt{r}$



Solution

Answer: C

Justification: At the top of the semicircle, where the velocity is lowest, the centripetal force must be greater than the force of gravity to prevent the ball from falling. Applying Newton's second law and the law of energy conservation:

$$\begin{cases} -mg + N = \dfrac{mv_{top}^2}{r} \Rightarrow \text{If } N = 0, v_{top} = \left(v_{top}\right)_{\min} = \sqrt{\dfrac{mgr}{m}} = \sqrt{gr} \\ \dfrac{mv_0^2}{2} = mgr + \dfrac{mv_{top}^2}{2} \end{cases}$$

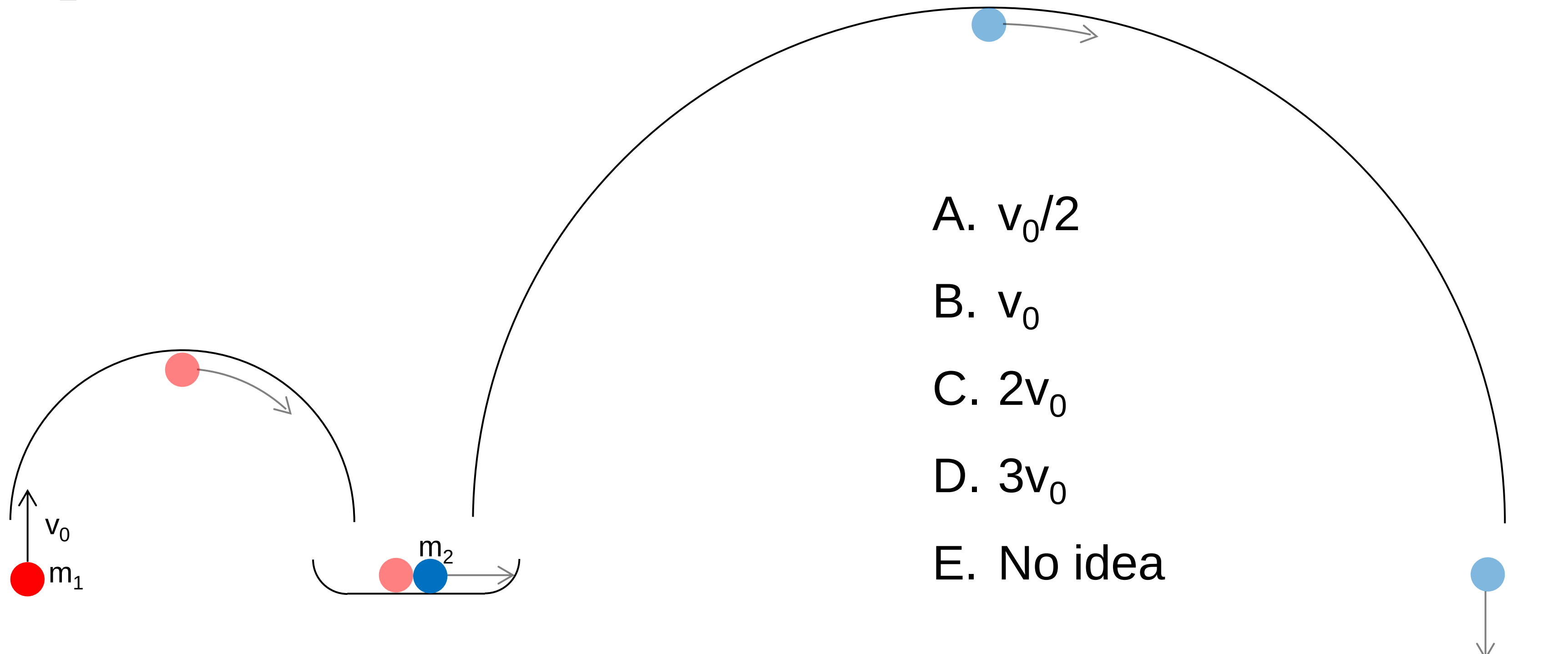
Newton's second law (positive x is directed upward)

Energy conservation law for the point at the top of the trajectory. Potential energy at the bottom of the trajectory – is zero.

$$\Rightarrow \frac{mv_0^2}{2} = mgr + \frac{m\left(\sqrt{gr}\right)^2}{2} \Rightarrow v_0^2 = 2gr + gr \Rightarrow v_0 > \sqrt{3gr}$$

Semicircles X

A contraption built out of two semicircular tubes is shown below. A ball with a mass of m_1 enters with an initial speed v_0 and elastically collides with a ball of mass m_2. The ball with mass m_2 then exits out the other end. What is the final speed of the ball with mass m_2 if $m_1 >> m_2$?



A. $v_0/2$
B. v_0
C. $2v_0$
D. $3v_0$
E. No idea

Solution

Answer: C

Justification: We know from question VIII that the speed of ball m_1 when it comes out of a semi-circle must be v_0. So when m_1 collides with m_2 it will be travelling at a speed of v_0. From the reference frame of m_1, m_2 travels towards it at a speed of v_0. Since m_1 is much heavier than m_2, m_1 will be almost unaffected by the collision (it will continue moving with the speed v_0), while m_2 will bounce off m_1 with a speed of $2v_0$, since its velocity has to be v0 relatively to ball m_1 - (See part I of this set, questions IV and V). Because m_2 returns to the same height after it has rounded the larger semicircle, it will travel at the same speed it had before traversing the semicircle, which is $2v_0$.

Of course you can solve it algebraically, by using the equations for elastic collisions of two balls of different masses as we did earlier…

Semicircles XI

$A.\ \sqrt{gr}$

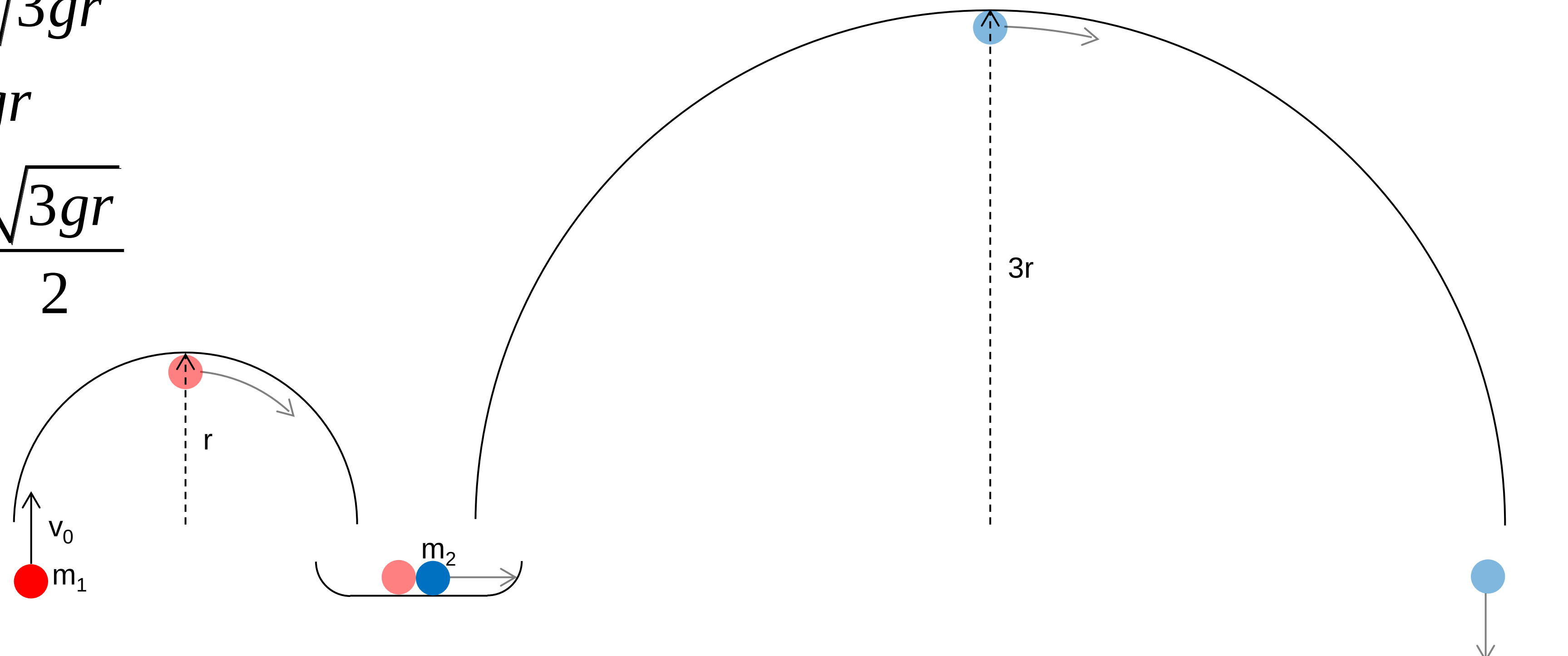
$B.\ \sqrt{2gr}$

$C.\ \sqrt{3gr}$

$D.\ gr$

$E.\ \frac{\sqrt{3gr}}{2}$

This is the same situation as question X, where $m_1 >> m_2$. What is the minimum value of v_0 required for m_2 to exit?



Solution

Answer: C

Justification: This question is somewhat trickier than the previous one. In the previous questions we found that the speed required for a ball to complete a semi-circle is $\sqrt{3gR}$. Therefore, for the small and big semi-circles, it is:

$$v_{small\_\min} = \sqrt{3gr}$$

$$v_{large\_\min} = \sqrt{3g3r} = \sqrt{3}v_{small\_\min} \approx 1.7v_{small\_\min} < 2v_{small\_\min} = 2v_0$$

Since m_2 travels with $2v_0$ and the minimum speed required to complete a large semi-circle is less than twice the speed needed to complete a small semi-circle, the minimum speed required for m_2 to exit is not the speed required for m_2 to travel around the semicircle, but rather the speed required for m_1 to complete the semi-circle and hit m_2, as m_2 can already traverse the semicircle at that speed. Thus, the answer to this question has the same answer as question X.

Semicircles XII

A. $2^{n-1}\sqrt{3gr}$

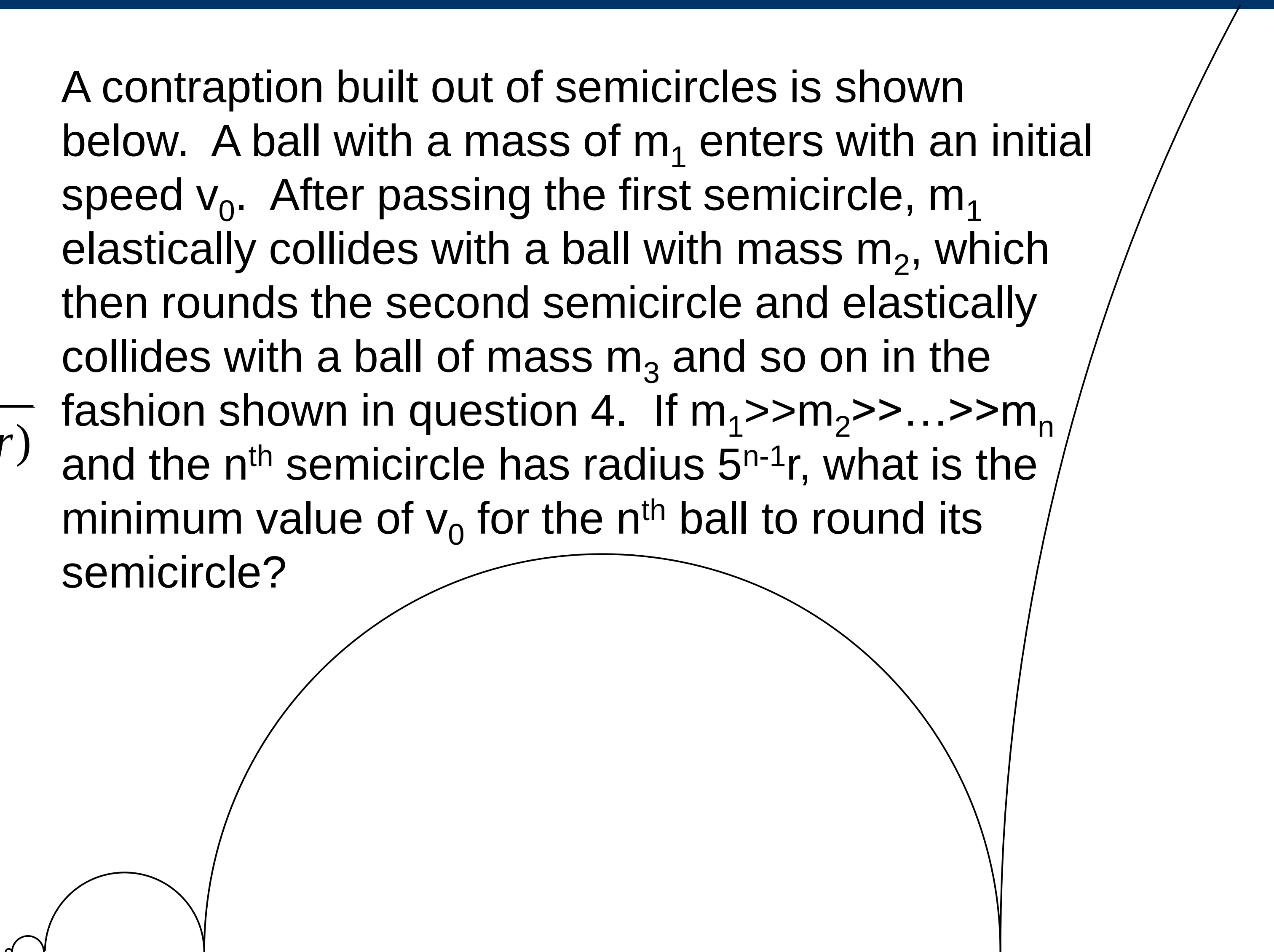
B. $\sqrt{3g(5^{n-1}r)}$

C. $\dfrac{\sqrt{3g(5^{n-1}r)}}{2^{n}}$

D. $2^{n-1}\sqrt{3g(5^{n-1}r)}$

E. $\dfrac{\sqrt{3g(5^{n-1}r)}}{2^{n-1}}$

A contraption built out of semicircles is shown below. A ball with a mass of m_1 enters with an initial speed v_0. After passing the first semicircle, m_1 elastically collides with a ball with mass m_2, which then rounds the second semicircle and elastically collides with a ball of mass m_3 and so on in the fashion shown in question 4. If $m_1 >> m_2 >> \ldots >> m_n$ and the n^{th} semicircle has radius $5^{n-1}r$, what is the minimum value of v_0 for the n^{th} ball to round its semicircle?



Solution

Answer: E

Justification: We know from questions III-V that m_2 travels at $2v_0$ when m_1 hits it with speed v_0. As $m_1 >> m_2 >> \ldots >> m_n$, m_3 travels at 2v if m_2 hits it with v. Therefore, m_n travels with a speed of $2^{n-1}v_0$. At the n^{th} semicircle, the radius is $5^{n-1}r$. From the identity we saw in question X, the minimum speed for the ball not to fall off a semi-circle is $v=\sqrt{(3gr)}$. In our case, it translates into:

$$2^{n-1}v_0 = \sqrt{3g(5^{n-1}r)}$$

$$v_0 = \frac{\sqrt{3g(5^{n-1}r)}}{2^{n-1}} = 5^{\frac{n-1}{2}} 2^{1-n} \sqrt{3gr}$$