a place of mind

FACULTY OF EDUCATION
Department of
Curriculum and Pedagogy

## Mathematics

## Numbers: Absolute Value of Functions

## Science and Mathematics Education Research Group

## Absolute Value of Functions I



## Absolute Values I

Which one of the following tables corresponds to $|f(x)|$, given the table of $f(x)$ at the right?
(Red numbers indicate values that were changed)

| $\mathbf{x}$ | $\mathbf{f}(\mathbf{x})$ |
| :---: | :---: |
| -4 | 11 |
| -2 | 0 |
| 0 | -3 |
| 2 | 6 |
| 4 | -8 |

B.

| $\mathbf{x}$ | $\|\mathbf{f}(\mathbf{x})\|$ |
| :---: | :---: |
| -4 | 11 |
| -2 | 0 |
| 0 | 3 |
| 2 | 6 |
| 4 | 8 |

C.

| $\mathbf{x}$ | $\|\mathbf{f}(\mathbf{x})\|$ |
| :---: | :---: |
| $\mathbf{4}$ | 11 |
| $\mathbf{2}$ | 0 |
| 0 | 3 |
| 2 | 6 |
| 4 | 8 |

D.

| $\mathbf{x}$ | $\|\mathbf{f}(\mathbf{x})\|$ |
| :---: | :---: |
| -4 | -11 |
| -2 | 0 |
| 0 | 3 |
| 2 | -6 |
| 4 | 8 |

## Solution

## Answer: B

Justification: The table of values of $|f(x)|$ are exactly the same as $f(x)$, except when $f(x)$ is negative. In this case, the value of $|f(x)|$ is made positive by multiplying $f(x)$ by -1 . This corresponds to table B , since every value under the $|f(x)|$ column is positive.

Table A and C are incorrect because the x-values were changed to positive. The domain of $|f(x)|$ is always the same as the domain of $f(x)$.

Table D is incorrect because all values of $f(x)$ where multiplied by -1 , not just the negative values.

## Absolute Values II

The graph of $f(x)$ is shown below. What is the graph of $|f(x)|$ ?

A.

C.

B.

D.


## Solution

## Answer: A

Justification: The graph of $|f(x)|$ must lie entirely above or on the x-axis since $|f(x)| \geq 0$. Only graph A satisfies this. Notice that the part of $f(x)$ that was negative has been reflected across the x-axis in order to become positive. This is the effect of multiplying $f(x)$ by -1 when it is negative.



## Solution Cont'd

Notice the graphs of $|f(x)|$ lie completely above or on the x axis. It is not possible for $|f(x)|$ to be less than 0 since negative values are turned positive. In this case, the range of $|f(x)|$ is $|f(x)| \geq 0$.



## Absolute Values III

Which of the following graphs corresponds to the function: $f(x)=|x|$
A.

C.

B.



## Solution

## Answer: B

Justification: The definition of the absolute value states that:

Notice how the absolute value function is composed of 2 lines, one for when $x \geq 0$ and one for when $x<0$.

Graph $D$ shows the graph of $f(x)=x$. What is the equation for graph A? (Write as a piecewise function.)

## Absolute Values IV

Which of the following piecewise functions corresponds to the absolute value graph shown below?

A.

$$
f(x)=\left\{\begin{array}{rr}
-2 x+4, & x \geq 2 \\
2 x+4, & x<2
\end{array}\right.
$$

B.

$$
f(x)=\left\{\begin{aligned}
2 x+4, & x \geq 2 \\
-2 x+4, & x<2
\end{aligned}\right.
$$

C.

$$
f(x)=\left\{\begin{aligned}
2 x-4, & x \geq 2 \\
-2 x+4, & x<2
\end{aligned}\right.
$$

D.
E.

$$
\begin{aligned}
& f(x)=\left\{\begin{array}{rr}
-2 x+4, & x \geq 2 \\
2 x-4, & x<2
\end{array}\right. \\
& f(x)=\left\{\begin{array}{rr}
2 x-4, & x \geq 2 \\
-\frac{1}{2} x+4, & x<2
\end{array}\right.
\end{aligned}
$$

## Solution

## Answer: C

Justification: $f(x)=\left\{\begin{array}{rr}2 x-4, & x \geq 2 \\ -2 x+4, & x<2\end{array}\right.$


The graph is best described by 2 different lines (one when $x>2$ and one when $x<2$ ). Find the equation of these lines individually and combine them into a single function in piecewise notation:

$$
f(x)=\left\{\begin{array}{rr}
2 x-4, & x \geq 2 \\
-2 x+4, & x<2
\end{array}\right.
$$

Notice that both the lines pass through the point $(2,0)$, so the $x=2$ case can be applied to either line.

## Alternative Solution

Answer: C $f(x)=\left\{\begin{array}{rr}2 x-4, & x \geq 2 \\ -2 x+4, & x<2\end{array}\right.$
Justification: First find the equation of the line when $x<2$. The slope is -2 and the $y$-intercept is 4 , so

$$
f(x)=-2 x+4 \quad \text { when } \quad x<2
$$

Notice that when $\mathrm{x} \geq 2, f(x)=-2 x+4$ is negative. Since the graph shown is an absolute value graph, we multiply $f(x)=-2 x+4$ by -1 when it is negative. Therefore

$$
f(x)=\left\{\begin{aligned}
2 x-4, & x \geq 2 \\
-2 x+4, & x<2
\end{aligned}\right.
$$

## Absolute Values V

What is the equation of the graph shown to the right, written with absolute values?
A. $f(x)=|2 x-4|$
B. $f(x)=|-2 x+4|$
C. $f(x)=|2 x+4|$
D. $f(x)=|-2 x-4|$

E. More than one of the above are correct

## Solution

Answer: E (both A and B are correct) Justification:

$$
f(x)=|2 x-4|=|-2 x+4|=\left\{\begin{array}{rr}
2 x-4, & x \geq 2 \\
-2 x+4, & x<2
\end{array}\right.
$$



Answers A and B are the same because the expressions inside the absolute values only differ by a factor of -1 .

$$
|f(x)|=|-f(x)|
$$

When $f(x)=|2 x-4|$, the graph is reflected across the $x$-axis when $\mathrm{x}<2$. When $f(x)=|-2 x+4|$, the graph is reflected across the x -axis when $x>2$.

## Absolute Values VI

Which of the following graphs corresponds to the equation:
A.


B.

$$
f(x)=\left|-\frac{1}{2} x+2\right|-2
$$




## Solution

## Answer: D

$$
f(x)=\left|-\frac{1}{2} x+2\right|-2
$$

Justification: Since 2 units are subtracted from the absolute value, the entire function is shifted 2 units down. Instead of negative values being reflected in the $x$-axis, they will be reflected across the line $y=-2$. This rules out answers $A$ and $B$.

In order to decide between C and D , try finding an ordered pair. If we plug in $\mathrm{x}=0$ into the function, we find that the graph should pass through ( 0,0 ):

$$
f(0)=\left|-\frac{1}{2}(0)+2\right|-2=0 \quad \text { Only } C \text { passes though }(0,0)
$$

(Another strategy to solve this problem is to first graph

$$
f(x)=\left|-\frac{1}{2} x+2\right| \text {, then shift it down by } 2 \text { units.) }
$$

## Absolute Values VII

The graph of $f(x)=\left|x^{2}-1\right|$ is shown below.
How many solutions are there to the equation $\left|x^{2}-1\right|=1$ ?

A. 4
B. 3
C. 2
D. 1
E. 0

## Solution

## Answer: B

Justification: The function $f(x)=\left|x^{2}-1\right|$ crosses the line $y=1$ three times. This means that there are 3 values of $x$ where

$$
f(x)=\left|x^{2}-1\right|=1
$$



The equation should have 3 solutions.

$$
\left|x^{2}-1\right|=1
$$

## Solution Cont'd

## Answer: B

Justification: The exact 3 solutions can be found by working with the equation. Divide the absolute value into 2 cases:

$$
\begin{array}{rlrlrl}
\left|x^{2}-1\right| & =1 & & \\
x^{2}-1 & =1 & \text { or } & x^{2}-1 & =-1 \\
x^{2} & =2 & & x^{2} & =0 \\
x & = \pm \sqrt{2} & x & =0
\end{array}
$$

The 3 solutions are $x=-\sqrt{2}, 0, \sqrt{2}$.

## Absolute Values VIII

The graph of $f(x)=\left|x^{2}-4 x-6\right|$ is shown below. For what values of $q$ does the equation $\left|x^{2}-4 x-6\right|=q$ have exactly 4 unique solutions?

A. $0<q<10$
B. $0 \leq q \leq 10$
C. $0<q \leq 10$
D. $0 \leq q<10$
E. $q \geq 10$

## Solution

## Answer: A

Justification: If the equation $\left|x^{2}-4 x-6\right|=q$ has 4 solutions, the line $y=q$ must cross $f(x)=\left|x^{2}-4 x-6\right|$ four times.


This happens when $q$ is between:

$$
0<q<10
$$

When $q=0$, the equation will only have 2 solutions.

When $q=10$, the equation will only have 3 solutions.

Notice that it is impossible for this equation to only have 1 solution.

