a place of mind

FACULTY OF EDUCATION

Department of
Curriculum and Pedagogy

Mathematics
Transformation of Functions

Science and Mathematics
Education Research Group

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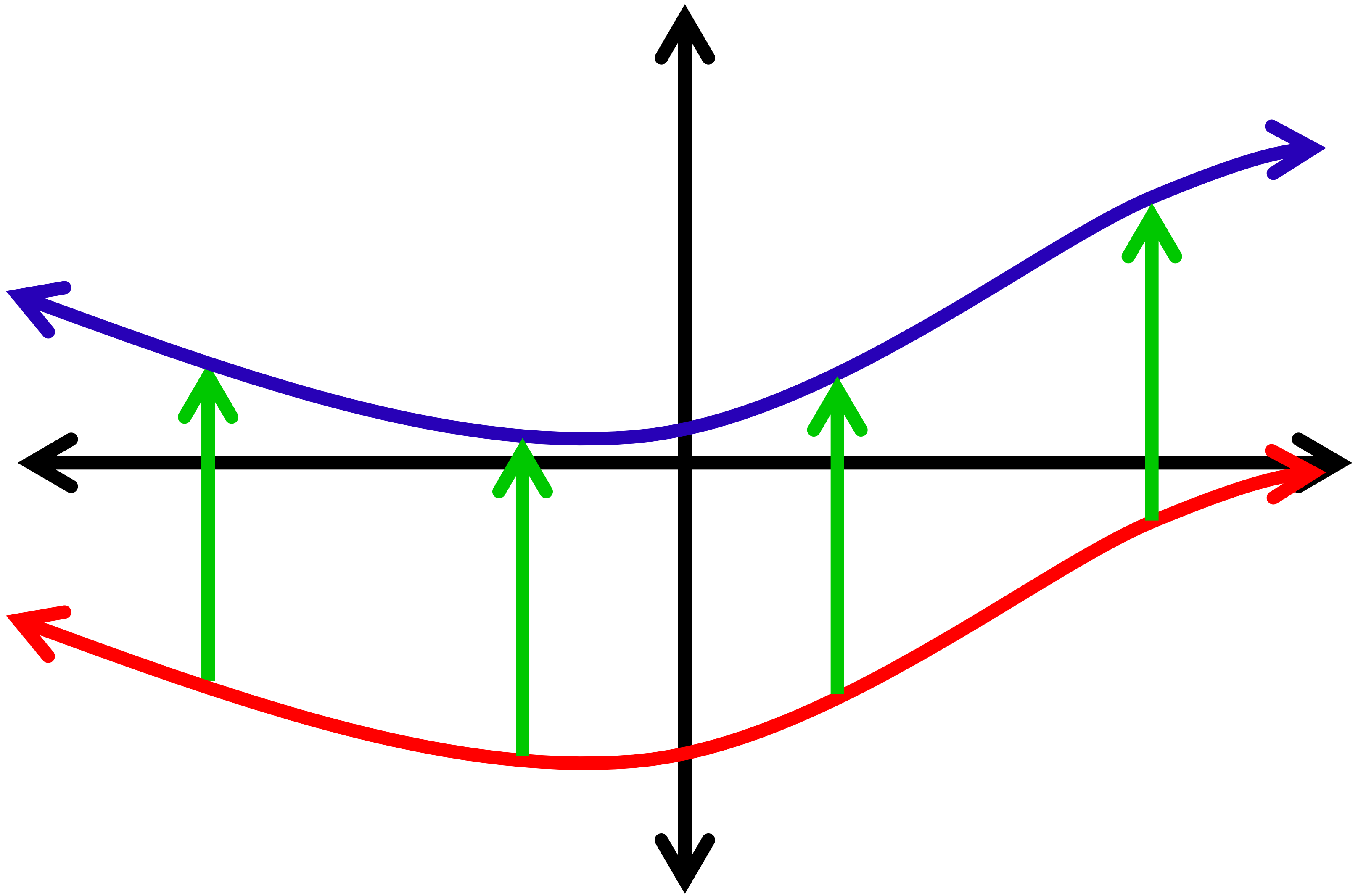
Transformation of Functions



Summary of Transformations

Vertical Translation	Horizontal Translation
$g(x) = f(x) + k$	$g(x) = f(x-k)$
k > 0, translate up	k > 0, translate right
k < 0 translate down	k < 0 translate left

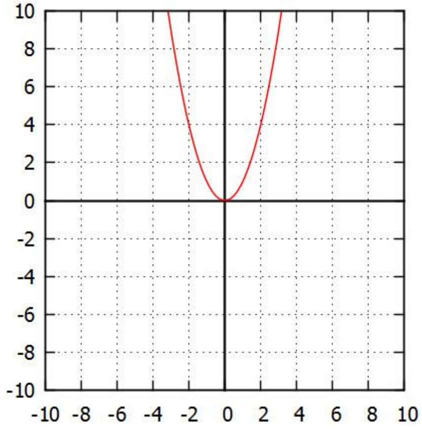
Reflection across x-axis	Reflection across y-axis
$g(x) = -f(x)$	$g(x) = f(-x)$
y-values change sign	x-values change sign

Vertical stretches	Horizontal stretches
$g(x) = k \cdot f(x)$	$g(x) = f\left(\frac{x}{k}\right)$
k > 1, expansion	k > 1, expansion
0 < k < 1 compression	0 < k < 1 compression

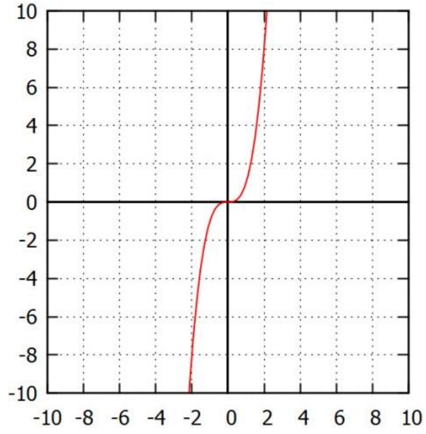
Standard Functions

You should be comfortable with sketching the following functions by hand:

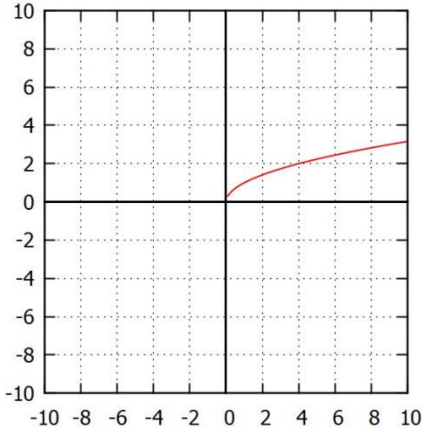
$f(x) = x^2$



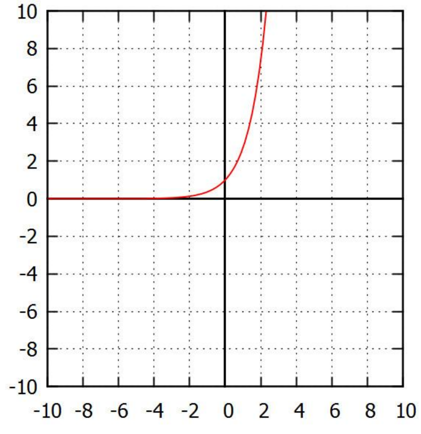
$f(x) = x^3$



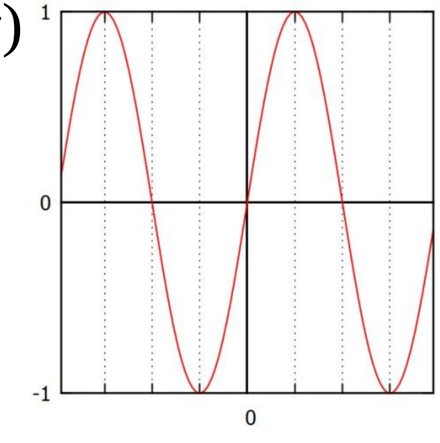
$f(x) = \sqrt{x}$



$f(x) = e^x$



$f(x) = \sin(x)$



Note on Terminology

This question set uses the following definitions for horizontal and vertical stretches:

Vertical stretches:

$$g(x) = k \cdot f(x)$$

k > 1, expansion

0 < k < 1 compression

Horizontal stretches:

$$g(x) = f\left(\frac{x}{k}\right)$$

k > 1, expansion

0 < k < 1 compression

For example, a vertical stretch by a factor of 0.5 is a compression, while a stretch by a factor of 2 is an expansion.

Other resources might say "a vertical compression by a factor of 2," implying that the reciprocal must be taken to determine the stretch factor.

Transformations on Functions

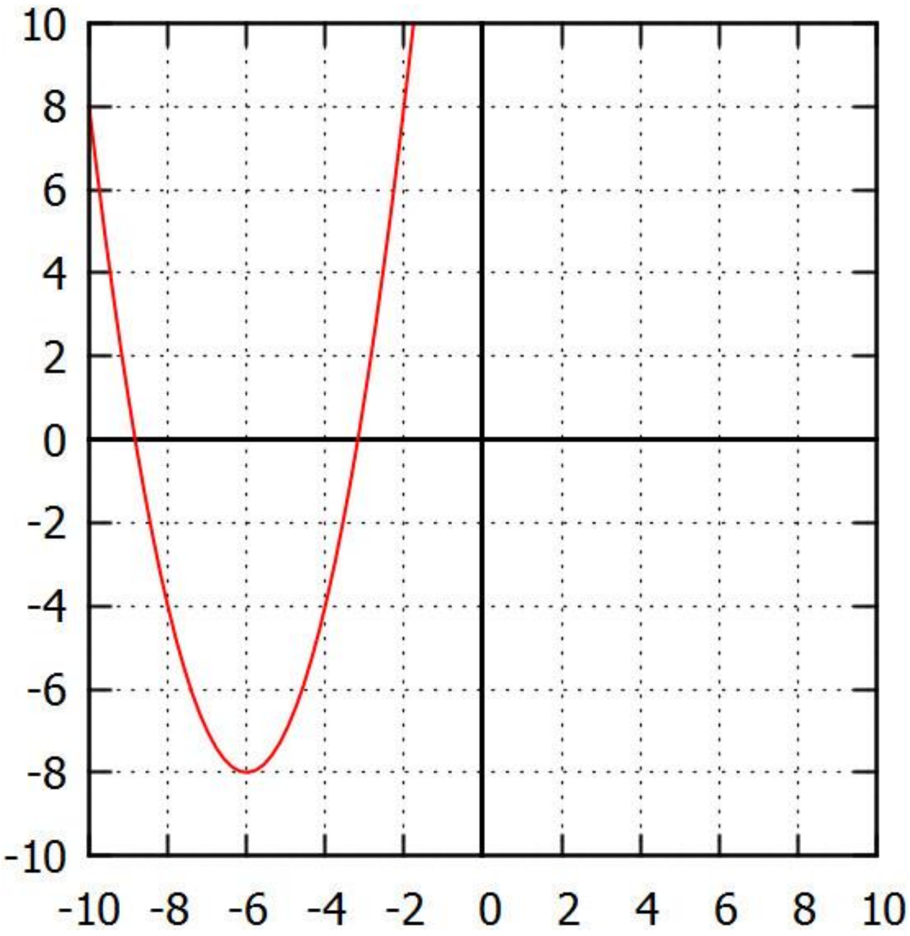
The graph to the right shows the function $f(x) = x^2$ after two transformations are applied to it. Which one of the following describe the correct transformations applied to f?

A. Horizontal translation -6 units, vertical translation -8 units

B. Horizontal translation 6 units, vertical translation 8 units

C. Horizontal translation 3 units, vertical translation 4 units

D. Horizontal translation -3 units, vertical translation -4 units



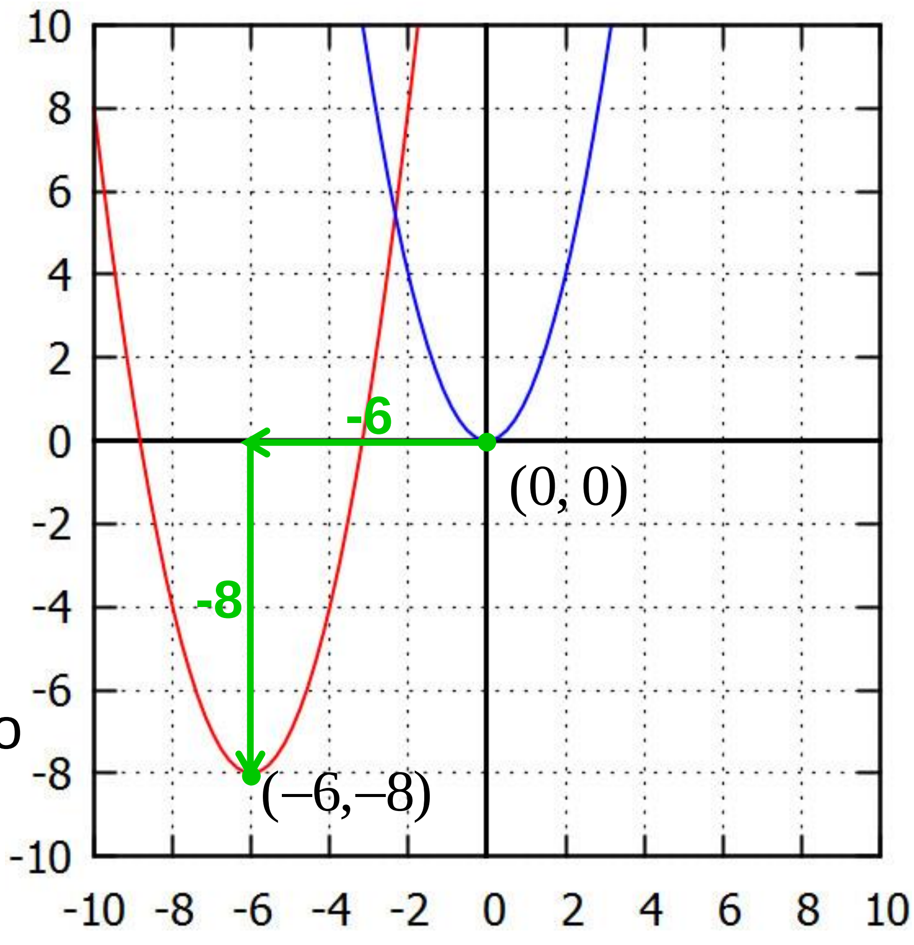
Solution

Answer: A

Justification: Consider the point (0,0) from $y = x^2$. It is easiest to determine how the vertex has been translated. The new vertex is located at (-6, -8).

Moving the function 6 units to the left corresponds to a *horizontal translation by -6 units*.

Moving 8 units down corresponds to *vertical translation by -8 units*.



Transformations on Functions II

The graph shown represents the equation $y = x^2$ after it has been translated 6 units to the left and 8 units down.

What is the equation of this function?

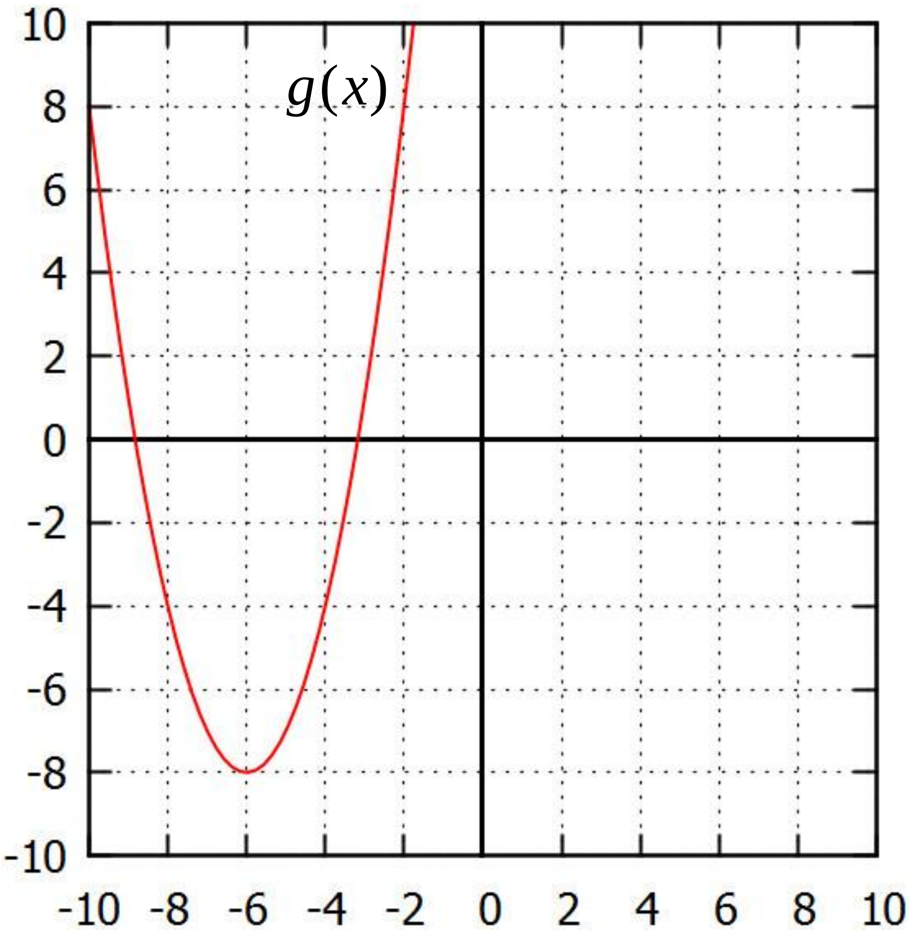
A. $g(x) = (x+6)^2 + 8$

B. $g(x) = (x+6)^2 - 8$

C. $g(x) = (x-6)^2 + 8$

D. $g(x) = (x-6)^2 - 8$

E. None of the above



Solution

Answer: B

Justification: We begin with the base equation of $f(x) = y = x^2$

Recall that for horizontal translations, we replace x with $x - k$. For vertical translations, we replace y with $y - k$.

Apply each substitution to the base equation to determine the final equation:

$y = x^2$	Base equation
$y = (x - (-6))^2$ $y = (x + 6)^2$	Replace x with $x - (-6)$; horizontal translation by -6 units (left)
$y - (-8) = (x + 6)^2$ $y = (x + 6)^2 - 8$	Replace y with $y - (-8)$; vertical translation by -8 units (down)
$g(x) = (x + 6)^2 - 8$	*Recall: the transformed function is labelled* g

Transformations on Functions III

The function $f(x) = \sqrt{x}$ is translated to form $g(x)$ (red).

What is the equation of $g(x)$?

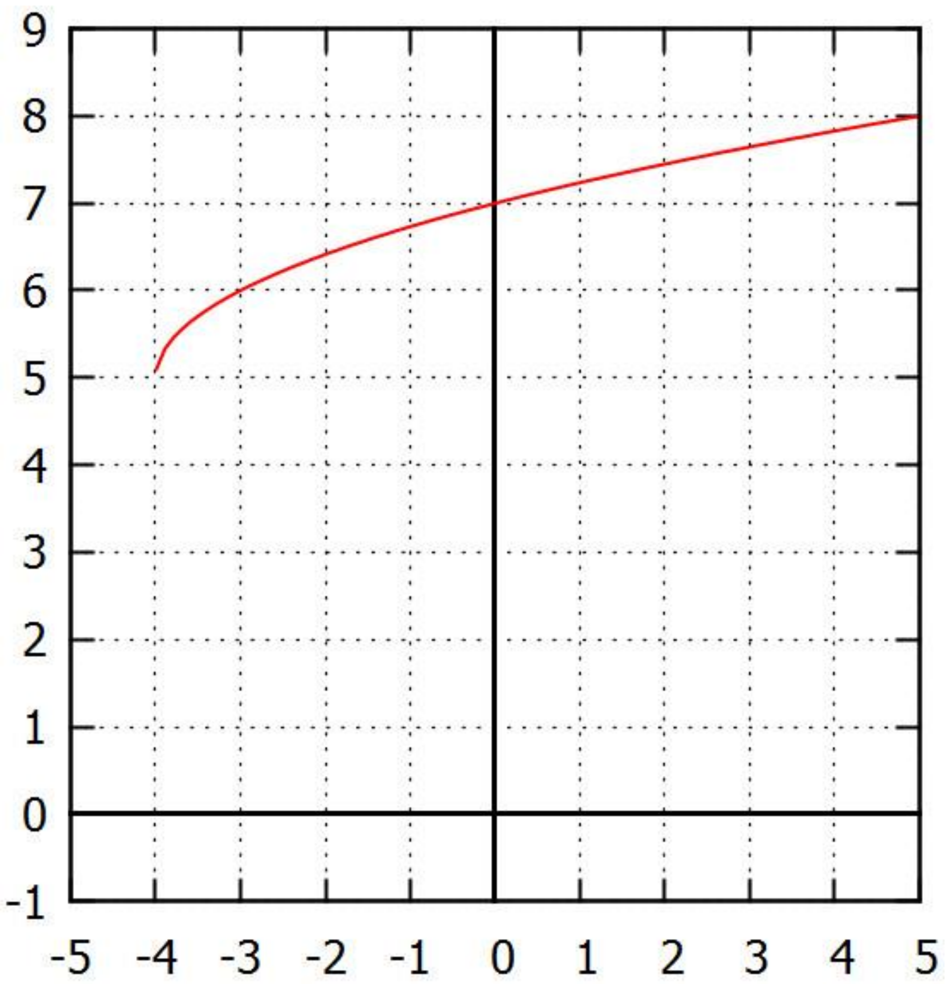
A. $g(x) = \sqrt{x+4} + 5$

B. $g(x) = \sqrt{x+5} + 4$

C. $g(x) = \sqrt{x-4} + 5$

D. $g(x) = \sqrt{x+5} - 4$

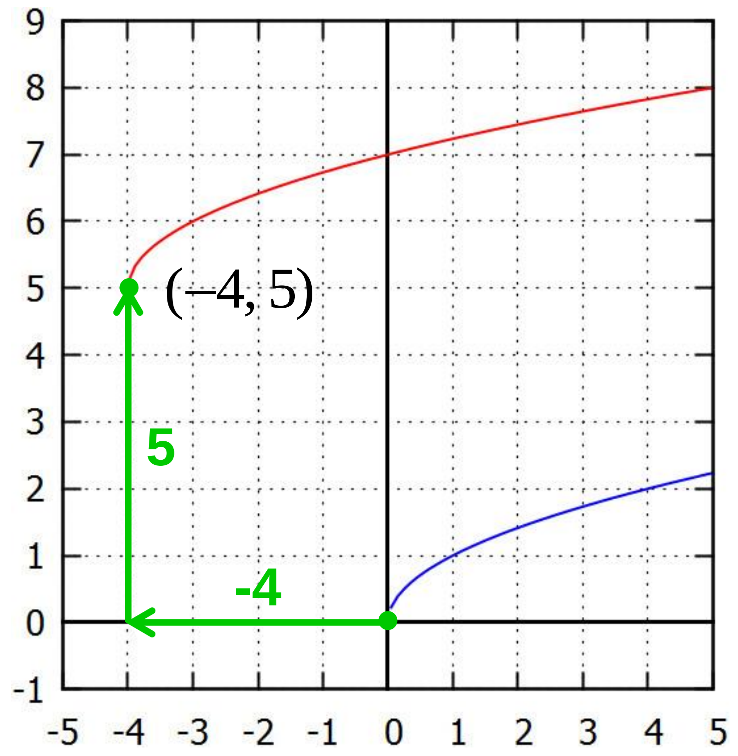
E. None of the above



Solution

Answer: A

Justification: Determine where the point (0, 0) in $f(x) = \sqrt{x}$ gets translated. This point is now located at (-4, 5). This is a horizontal translation by -4 units (left), and vertical translation by 5 units (up). Note: The order that the translations are applied does not matter.



$f(x) = y = \sqrt{x}$	Base equation
$y = \sqrt{x-(-4)}$	Replace x with $x-(-4)$
$y-5 = \sqrt{x+4}$	Replace y with $y-5$
$g(x) = \sqrt{x+4}+5$	

Transformations on Functions IV

The function $f(x) = x^3$ is first reflected in the x-axis, and then translated as shown.

What is the equation of the new function, $g(x)$?

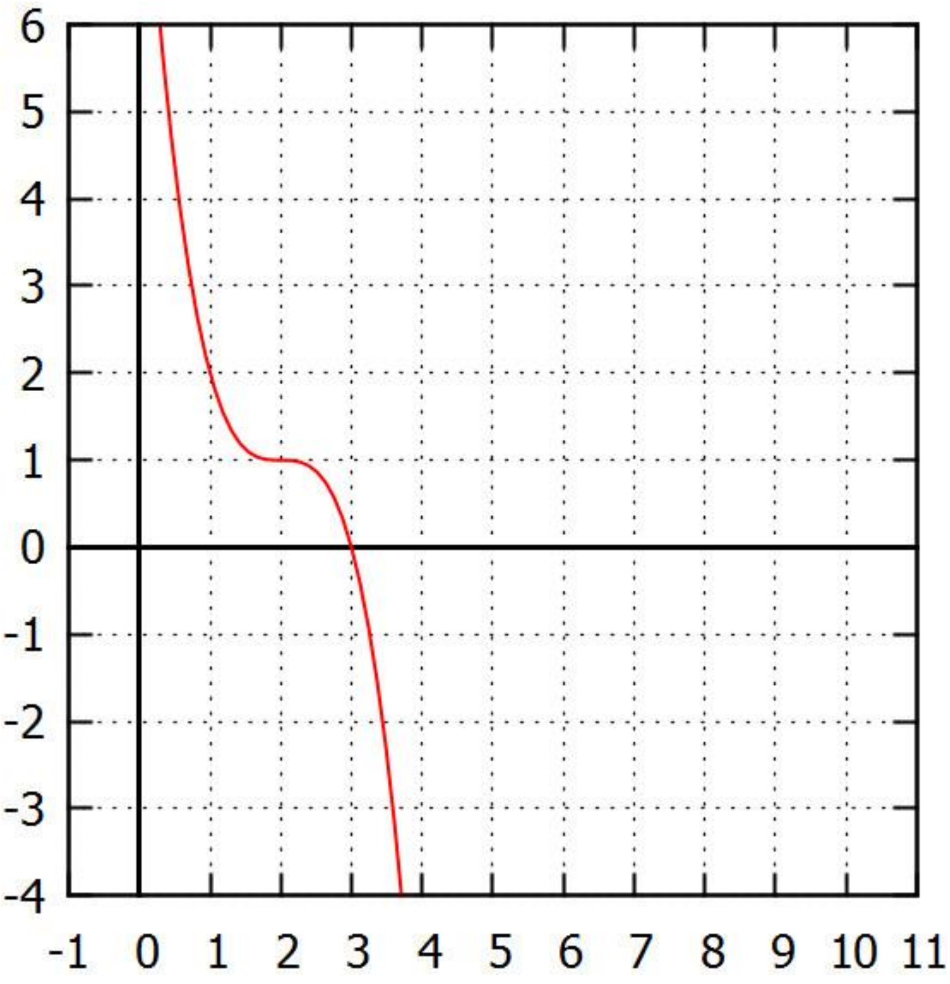
A. $g(x) = (x-2)^3 + 1$

B. $g(x) = -(x-2)^3 - 1$

C. $g(x) = -(x-2)^3 + 1$

D. $g(x) = (-x-2)^3 + 1$

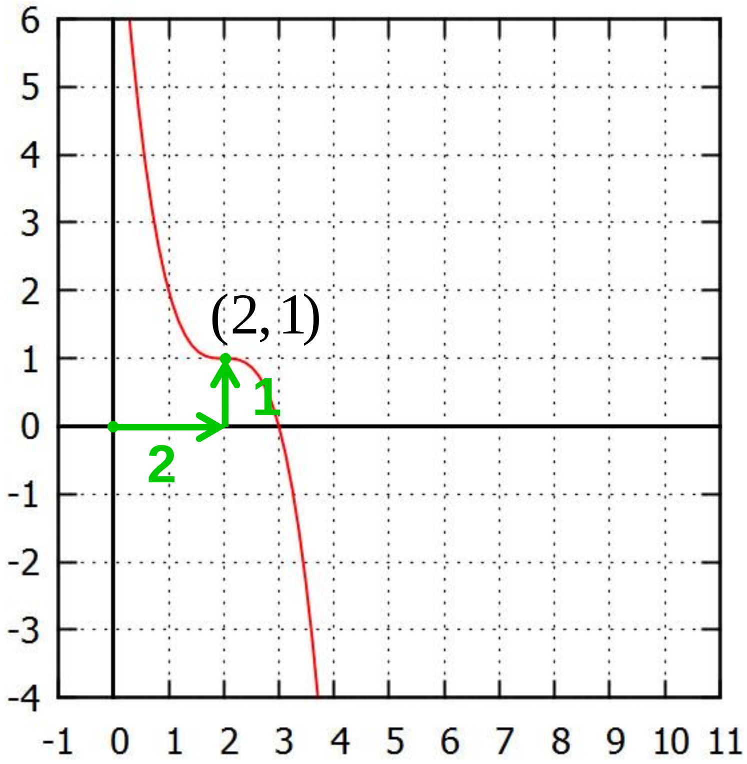
E. $g(x) = (-x+2)^3 + 1$



Solution

Answer: C

Justification: Recall that reflections across the x-axis require replacing y with -y. Use the point (0, 0) on the graph $y = x^3$ in order to determine how cubic functions are translated.



Perform the substitutions:

$f(x) = y = x^3$	Base equation
$-y = x^3$	Replace y with $-y$
$y = -x^3$	
$y = -(x-2)^3$	Replace x with $x-2$; translate 2 units right
$y - 1 = -(x-2)^3$	Replace y with $y-1$; translate 1 unit up
$g(x) = -(x-2)^3 + 1$	

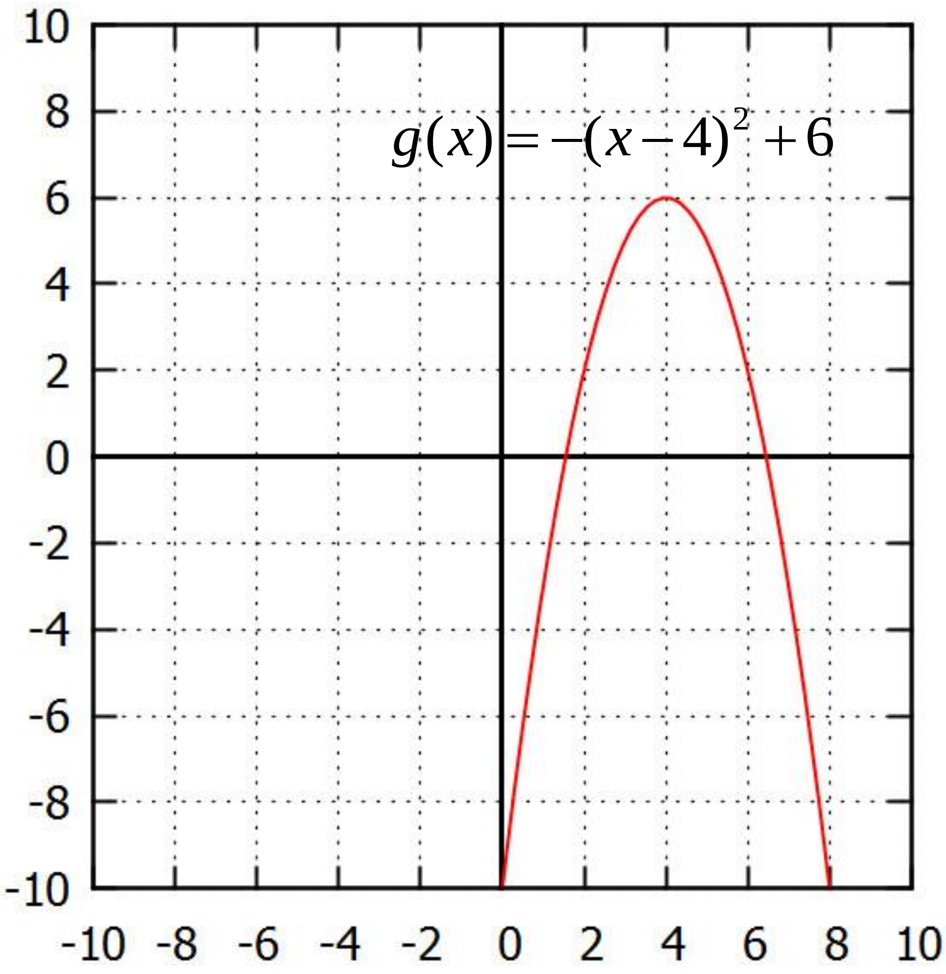
Transformations on Functions V

The function $f(x) = x^2$ is first reflected in the x-axis and then translated 4 units right and 6 units up to give $g(x)$.

Would the resulting function be different if it were translated first, and then reflected in the x-axis?

A. Yes

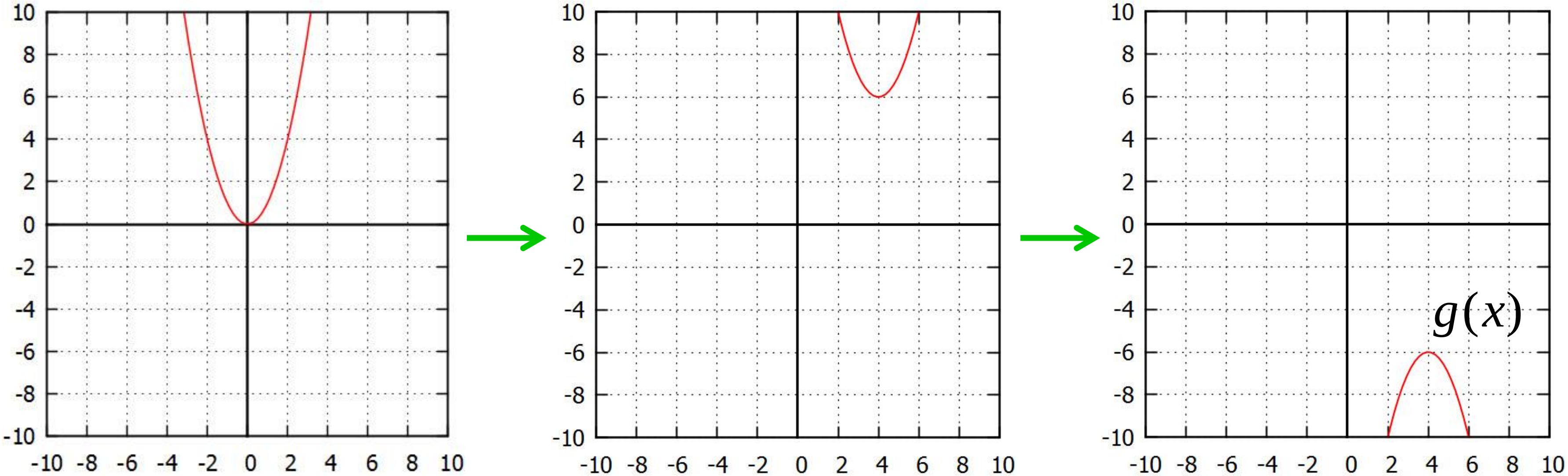
B. No



Solution

Answer: A

Justification: Draw the graph of $g(x)$ if translations were done first before the reflection and compare with the given graph:



Base graph

Translate 4 right; 6 up

Reflect across x-axis

Instead of finishing 6 units up, $g(x)$ was translated 6 units down.

Alternative Solution

Answer: A

Justification: Determine the equation of $g(x)$ if the translation substitutions are done first before the reflection.

$f(x) = y = x^2$	Base equation
$y = (x-4)^2$	Replace x with $x-4$; translate 4 units right
$y-6 = (x-4)^2$	Replace y with $y-6$; translate 6 units up
$-y = (x-4)^2 + 6$	Replace y with $-y$; reflection in the x-axis
$g(x) = -(x-4)^2 - 6$	

Since the reflection was done after translating 6 units up, the negative sign from the reflection also changes the sign of the vertical translation. Compare this to the original equation:

$$g(x) = -(x-4)^2 + 6$$

Transformations on Functions VI

The function $f(x) = \ln(x)$ is reflected in the y-axis, and then translated left 2 units and up 4 units. Which of the following sets of transformations will result in the same function as the transformations outlined above?

A. Translate up 4 units, translate left 2 units, reflect in y-axis

B. Translate up 4 units, translate right 2 units, reflect in y-axis

C. Translate down 4 units, translate left 2 units, reflect in y-axis

D. Translate down 4 units, translate right 2 units, reflect in y-axis

E. More than 1 of the above are correct

(Notice that the reflection is done after the translations)

Solution

Answer: B   Translate up 4 units, translate right 2 units, reflect in y-axis

Justification: Notice that when a y-axis reflection is done at the after a horizontal translation, the direction of the translation also gets reflected.

Example:

(1, 0) —4 units up→ (1, 4) —2 units right→ (3, 4) —Reflect y-axis→ (-3, 4)

(1, 0) —Reflect y-axis→ (-1, 0) —4 units up→ (-1, 4) —2 units left→ (-3, 4)

The next slide shows how making the transformation substitutions into the equations results in the same function.

Solution Continued

Answer: B Translate up 4 units, translate right 2 units, reflect in y-axis

Justification: First find the equation of the function we are trying to match: $f(x) = y = \ln(x)$ Base equation

$y = \ln(-x)$ Replace x with $-x$; reflect in y-axis

$y = \ln[-(x-(-2))]$ Replace x with $x-(-2)$; 2 units left

$y-4 = \ln[-(x+2)]$ Replace y with $y-4$; 4 units up

$g(x) = \ln[-x-2]+4$

If the reflection is done at the end:

$y = \ln(x)$

$y-4 = \ln(x)$ Replace y with $y-4$; 4 units up

$y = \ln(x-2)+4$ Replace x with $x-2$; 2 units right

$y = \ln[(-x)-2]+4$ Replace x with $-x$; reflect in y-axis

$y = \ln[-x-2]+4$

Transformations on Functions VII

The graph $f(x)=\sqrt{1-x^2}$ is shown in red. It is then reflected in the x-axis, reflected in the y-axis, and translated to the right by 1 unit. Which graph represents $f(x)$ after these transformations?

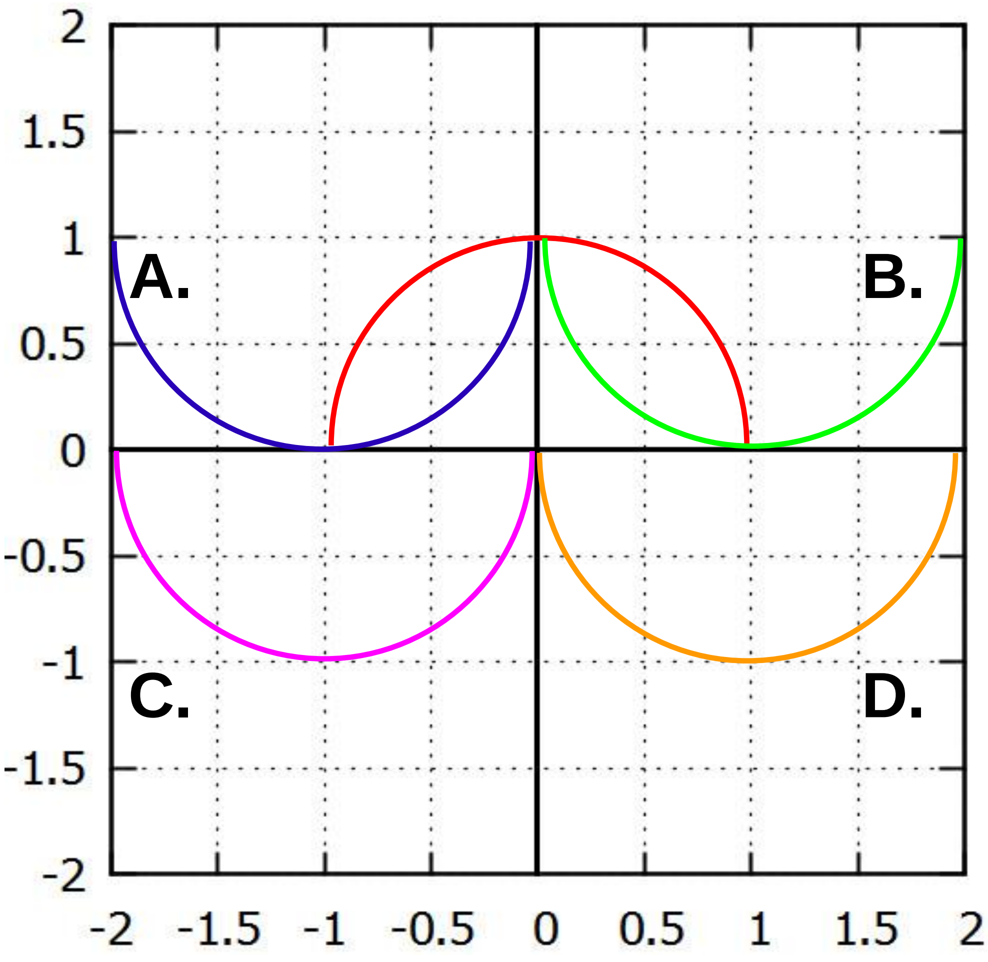
A. Blue graph

B. Green graph

C. Purple graph

D. Orange graph

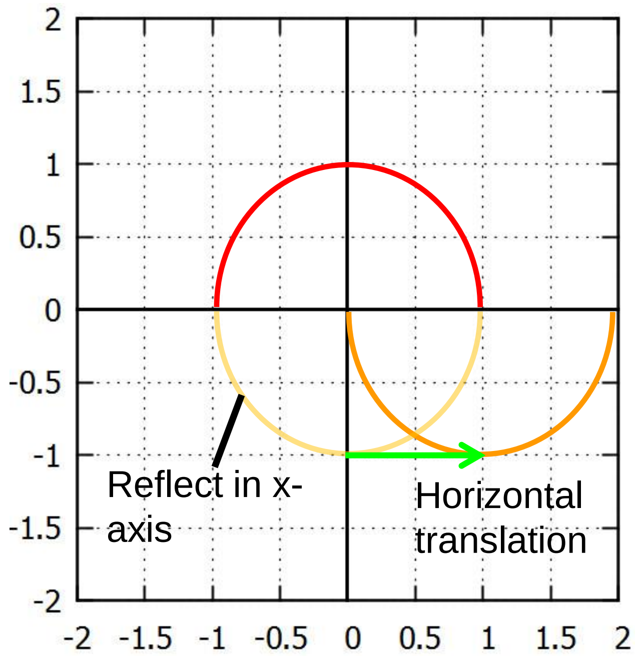
E. None of the graphs



Solution

Answer: D

Justification: The transformations can be performed as shown in the graph below. Notice that reflection in y-axis has no effect on the graph, since the graph has a line of symmetry across the y-axis.



The factor of -1 from the reflection in y-axis is inside a square, and therefore does not change the function. All the equations below are equivalent:

$$g(x) = -\sqrt{1-(-(x-1))^2}$$
$$= -\sqrt{1-(-x+1)^2}$$
$$= -\sqrt{1-(x-1)^2}$$

Transformations on Functions VIII

The function $f(x) = x^3 - x^2 + x - 1$ is reflected in the x-axis, and then reflected in the y-axis. What is the equation of the resulting function, $g(x)$?

A. $g(x) = x^3 - x^2 + x - 1$

B. $g(x) = -x^3 + x^2 - x + 1$

C. $g(x) = x^3 - x^2 + x + 1$

D. $g(x) = -x^3 - x^2 - x + 1$

E. $g(x) = x^3 + x^2 + x + 1$

Solution

Answer: E

Justification: Perform the transformation substitutions:

$f(x) = y = x^3 - x^2 + x - 1$	Base equation
$-y = x^3 - x^2 + x - 1$	Replace y with $-y$; reflect in x-axis
$y = -x^3 + x^2 - x + 1$	Move the negative from left to right
$y = -(-x)^3 + (-x)^2 - (-x) + 1$	Replace x with $-x$; reflect in y-axis
$g(x) = x^3 + x^2 + x + 1$	

Remember than $(-x)^n$ is positive when *n* is even, negative when *n* is odd.

Transformations on Functions IX

The function $f(x) = x^3$ is expanded horizontally by a factor of 2. It is then translated horizontally by -2 units. What is the equation of this function?

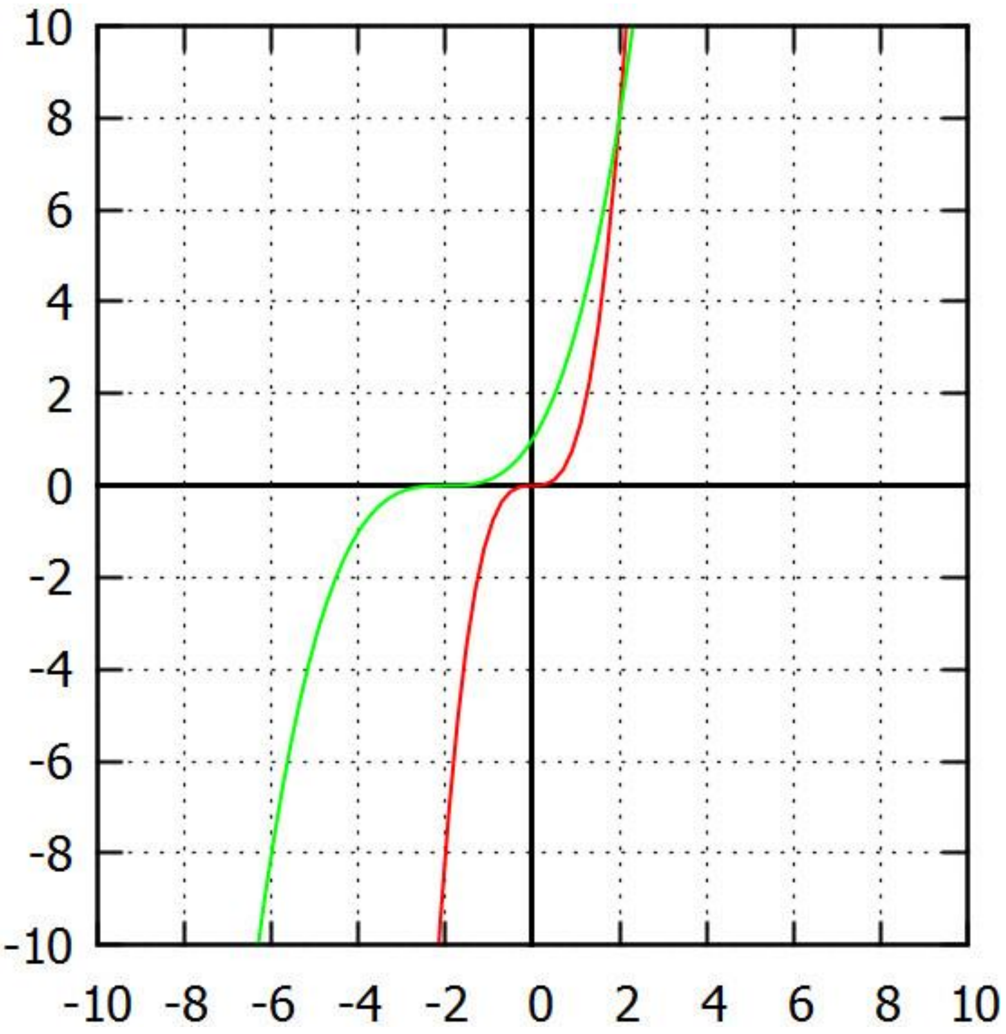
A. $g(x) = 8(x+2)^3$

B. $g(x) = \frac{1}{8}(x+2)^3$

C. $g(x) = (2x+2)^3$

D. $g(x) = \left(\frac{1}{2}x+2\right)^3$

E. $g(x) = (2x+4)^3$



Solution

Answer: B

Justification: Recall that for horizontal stretches by a factor of *k*, we replace x with $\frac{x}{k}$.

$f(x) = y = x^3$	Base equation
$y = \left(\frac{x}{2}\right)^3$	Replace x with $\frac{x}{2}$; horizontal stretch by 2
$y = \left(\frac{x-(-2)}{2}\right)^3$	Replace x with $x-(-2)$; shift left by 2
$g(x) = \frac{(x+2)^3}{8}$	We can then take the denominator out by cubing it

Transformations on Functions X

The function $f(x) = \sqrt{4-(x+1)^2}$ is shown in red. It is then stretched vertically by 2, and horizontally by 0.5. Which is the correct resulting graph?

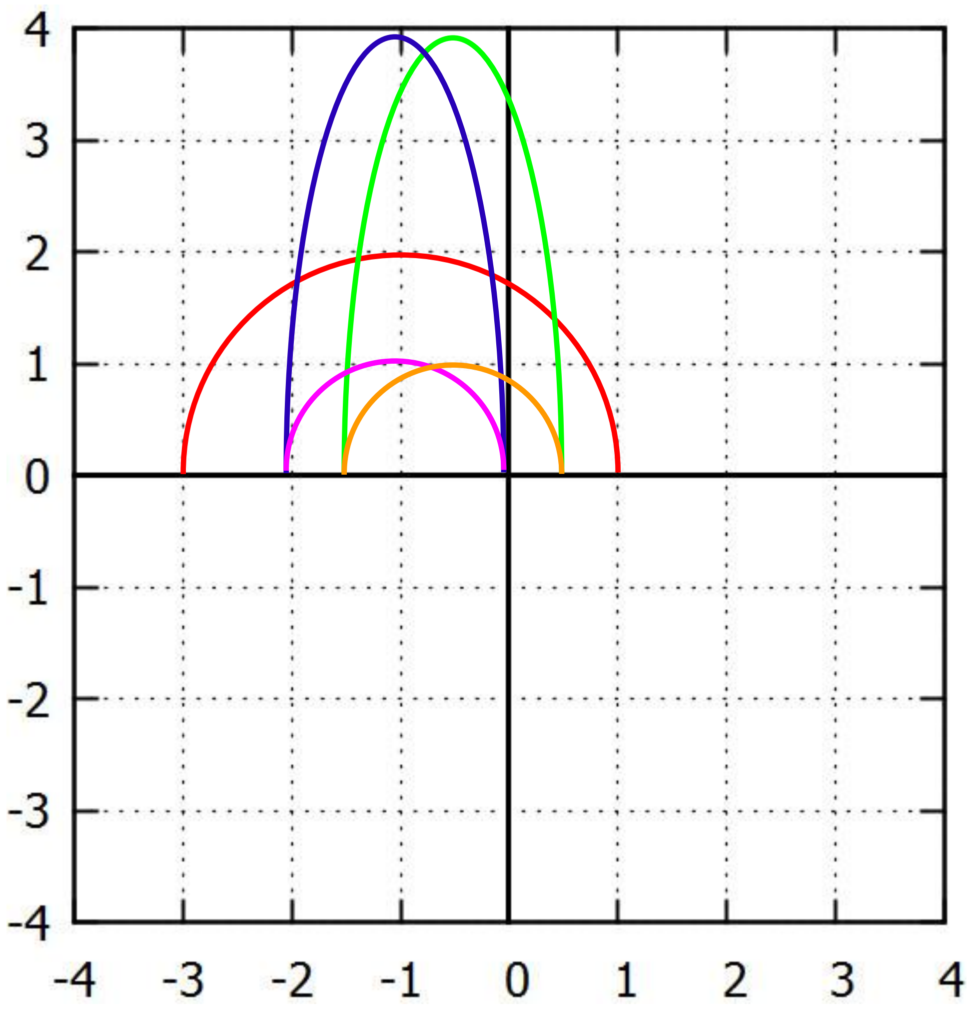
A. Blue graph

B. Green graph

C. Purple graph

D. Orange graph

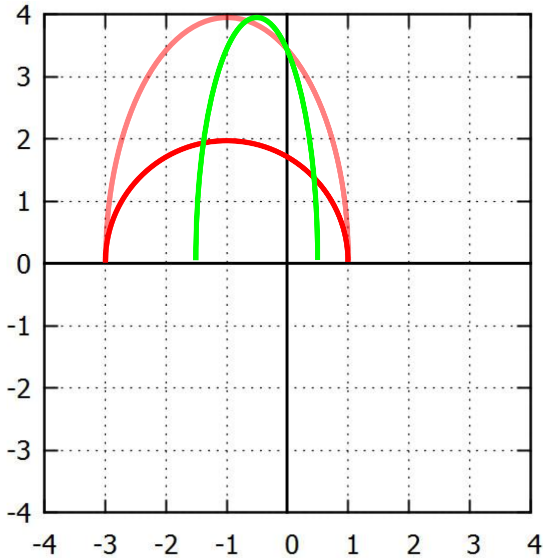
E. None of the graphs



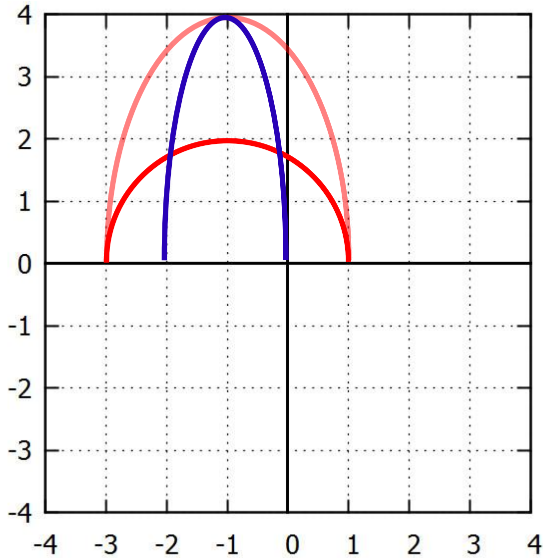
Solution

Answer: B

Justification: A vertical stretch by 2 multiplies all y-values by 2. A horizontal stretch by 0.5 divides all x-values by 2.



Correct: Note how (-3, 0) moves to (-1.5, 0) and (1, 0) moves to (0.5, 0). The graph is scaled correctly.



Incorrect: Even though the graph is scaled correctly, notice how the point (1, 0) incorrectly moves to (0, 0)

Transformations on Functions XI

The two functions $f(x)$ and $g(x) = a \cdot f(bx)$ are shown to the right.

What are values of *a* and *b*?

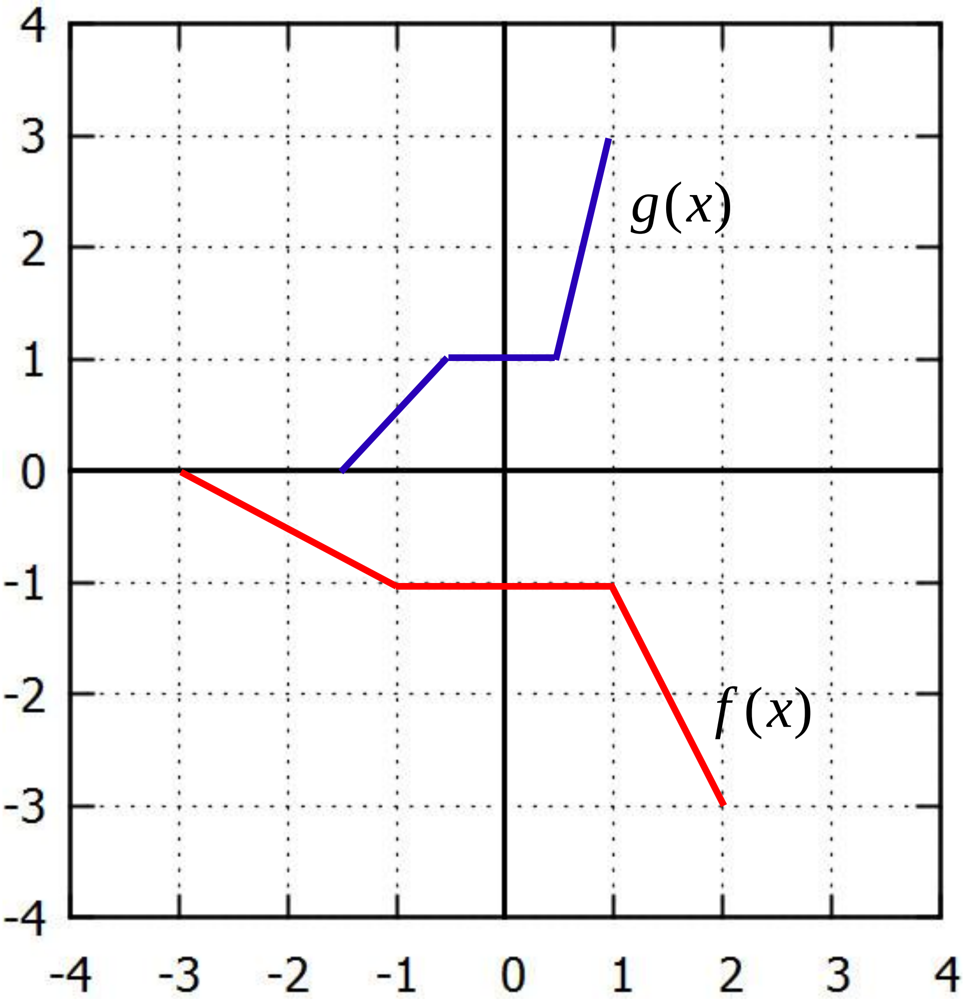
A. $a = -1, \quad b = 2$

B. $a = -1, \quad b = \frac{1}{2}$

C. $a = 2, \quad b = -1$

D. $a = \frac{1}{2}, \quad b = -1$

E. $a = -2, \quad b = 1$

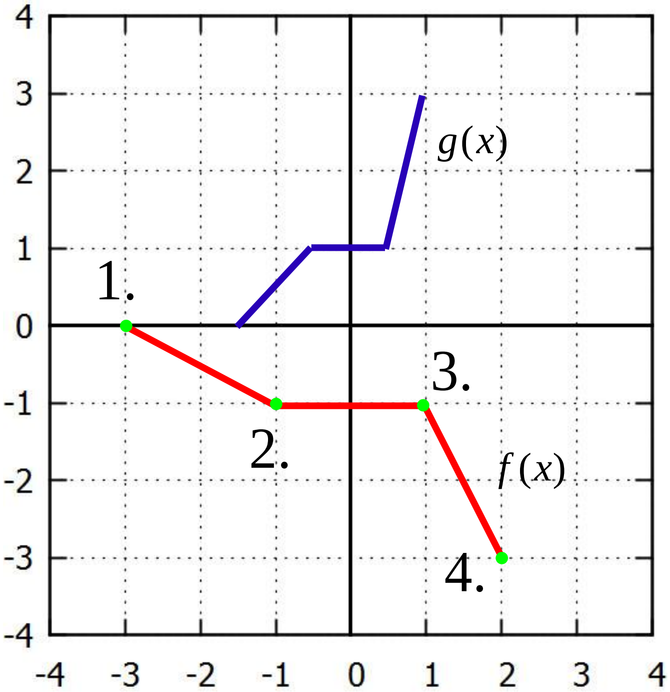


Solution

Answer: A

Justification: Pick a few test points on $f(x)$ and note how they are transformed:

1. $(-3, 0) \rightarrow (-1.5, 0)$
2. $(-1, -1) \rightarrow (0.5, 1)$
3. $(1, -1) \rightarrow (0.5, 1)$
4. $(2, -3) \rightarrow (1, 3)$



Since the x-coordinates are reduced by a half and the y-coordinates change signs, the transformations are:

Reflection across x-axis

Horizontal compression by 0.5.

$$g(x) = -1 \cdot f(2x)$$

$$a = -1, \quad b = 2$$

Transformations on Functions XII

The point P(a, b) is on the function $f(x)$. If $g(x) = 2f(1-x)+3$, where is point P on $g(x)$?

A. $(-a-1, 2b+3)$

B. $(-a-1, -2b+3)$

C. $(-a+1, 2b+3)$

D. $(-a+1, -2b+3)$

E. None of the above

Solution

Answer: C

Justification: It may be helpful to write $g(x)$ as: $g(x) = 2f(1-x)+3$

$$= 2f\big(-(x-1)\big)+3$$

Work backwards from the transformation substitutions to determine the transformations applied to $f(x)$:

Vertical expansion by 2	$g(x) = 2f(x)$
Reflection in y-axis	$g(x) = f(-x)$
Translate 3 units up	$g(x) = f(x)+3$
Translate 1 unit right	$g(x) = f(x-1)$

$$g(x) = 2f(x)$$
$$= 2f(-x)$$
$$= 2f(-x)+3$$
$$= 2f(-(x-1)+3$$

The point P(a, b) will then be located at $(-a+1, 2b+3)$.