a place of mind

FACULTY OF EDUCATION

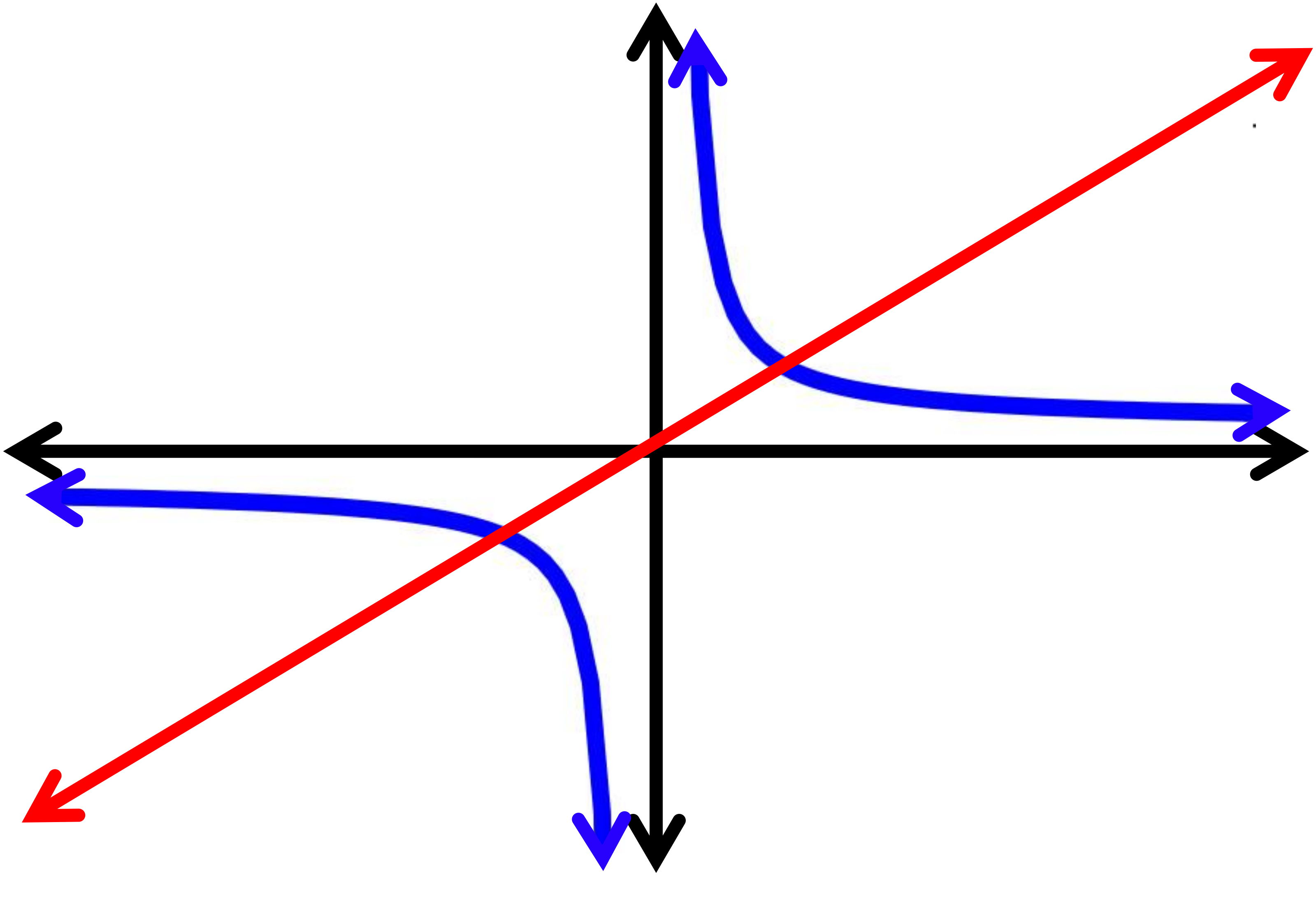
Department of
Curriculum and Pedagogy

Mathematics
Reciprocal Functions

Science and Mathematics
Education Research Group

Supported by UBC Teaching and Learning Enhancement Fund 2012-2014

Reciprocal Functions



Reciprocal Functions

This question set deals with functions in the form:

$$g(x) = \frac{1}{f(x)} = \left(f(x)\right)^{-1}$$

Given the function $f(x)$, we will analyze the shape of the graph of $g(x)$.

Notice the difference between the reciprocal of a function, and the inverse of a function:

Reciprocal:

$$g(x) = \left(f(x)\right)^{-1}$$

$$(a,b) \to (a, \frac{1}{b})$$

Inverse:

$$g(x) = f^{-1}(x)$$

$$(a,b) \to (b,a)$$

The <u>reciprocal</u> and the <u>inverse</u> of a function are not the same.

Reciprocal Functions I

Consider the value of $\frac{1}{p}$ (the reciprocal of p) , where $p > 0$.

Which of the following correctly describes the approximate value of $\frac{1}{p}$ for varying values of p?

	p is very large (for example, p = 1000)	**p is very small (for example, p = 0.0001)**
A.	$\frac{1}{p}$ is large	$\frac{1}{p}$ is large
B.	$\frac{1}{p}$ is close to 0	$\frac{1}{p}$ is large
C.	$\frac{1}{p}$ is large	$\frac{1}{p}$ is close to 0
D.	$\frac{1}{p}$ is close to 0	$\frac{1}{p}$ is close to 0
E.	$\frac{1}{p} = \infty$	$\frac{1}{p} = 0$

Solution

Answer: B

Justification: Fractions with large denominators are smaller than fractions with small denominators (for equal positive numerators).

Consider when $p = 1000$. Its reciprocal is a very small positive number:

$$\frac{1}{p} = \frac{1}{1000} = 0.001$$

As p becomes larger, its reciprocal becomes smaller.

Now consider when $p = 0.001$. Its reciprocal is a number much greater than 1:

$$\frac{1}{p} = \frac{1}{0.001} = \frac{1}{\frac{1}{1000}} = 1000$$

As p becomes smaller, its reciprocal becomes larger.

Reciprocal Functions II

Consider the function $g(x) = \left(f(x)\right)^{-1}$.

When $0 < f(x) < 1$, what are the possible values for $g(x)$?

A. $g(x) > 0$

B. $g(x) > 1$

C. $0 < g(x) < 1$

D. $g(x) < 0$

E. $g(x) < 1$

Exponent laws:

$$a^{-x} = \frac{1}{a^x}$$

$$g(x) = \left(f(x)\right)^{-1}$$

$$g(x) = \frac{1}{f(x)}$$

Press for hint

Solution

Answer: B

Justification: Try choosing a value for $f(x)$, such as $f(x)=\frac{1}{2}$.

In this case, $g(x)=\left(f(x)\right)^{-1}=\dfrac{1}{f(x)}=\dfrac{1}{\frac{1}{2}}=2$

Since $g(x)=2$, which is greater than 1, we can rule out C, D, and E.

We must check if it is possible for $g(x)$ to be between 0 and 1 in order to choose between answers A and B.

Notice that $g(x)={}^{1}\!/_{f(x)}$ is only between 0 and 1 when the numerator is smaller than the denominator. This is not possible, because the denominator $(0<f(x)<1)$ is never greater than 1.

Therefore, $g(x)>1$ when $0<f(x)<1$.

Reciprocal Functions III

Consider a function $f(x)$ and its reciprocal function $g(x) = \left(f(x)\right)^{-1}$. Suppose that at $x = p$, $f(p) = g(p)$. What are all the possible values for $f(p)$?

A. $f(p) = 1$

B. $f(p) = 0$

C. $f(p) = -1$

D. $f(p) = \pm 1$

E. $f(p) = 0, \pm 1$

Solution

Answer: D

Justification: This question asks to find the values of $f(p)$ where $f(p) = \frac{1}{f(p)}$. We must find a value that is equal to its reciprocal.

We can get the answer directly by solving the above equation for $f(p)$:

$$f(p) = \frac{1}{f(p)} \quad \Rightarrow \quad \left(f(p)\right)^2 = 1 \quad \Rightarrow \quad f(p) = \pm 1$$

Notice: $1 = \frac{1}{1}, \quad -1 = \frac{1}{-1}$

If we tried to plug in $f(p) = 0$, we get $g(p) = 1/0$ which is undefined.

Conclusion: Points where $f(x) = \pm 1$ are also points on the reciprocal of f. Points where $f(x) = 0$ form vertical asymptotes on the reciprocal function.

Reciprocal Functions IV

Let $g(x) = \dfrac{1}{f(x)}$.

Which of the following correctly describes $g(x)$ when $f(x) > 0$ and when $f(x) < 0$?

	$f(x) > 0$	$f(x) < 0$
A.	$g(x) < 0$	$g(x) < 0$
B.	$g(x) < 0$	$g(x) > 0$
C.	$g(x) > 0$	$g(x) < 0$
D.	$g(x) > 0$	$g(x) > 0$
E.	$0 < g(x) < 1$	$-1 < g(x) < 0$

Solution

Answer: C

Justification: When $f(x) > 0$, $1/f(x)$ must also be greater than zero. The numerator is positive, and the denominator is positive. A positive number divided by a positive number is positive.

When $f(x) < 0$, $1/f(x)$ must also be less than zero. The numerator is positive but the denominator is negative. A positive number divided by a negative number is negative.

Conclusion: If $f(x)$ is above the x-axis, $g(x)$ is also above the x-axis. When $f(x)$ is below the x-axis, $g(x)$ is also below the x-axis.

Reciprocal Functions V

If the point (a,b), where $b \neq 0$, lies on the graph of f, what point lies on the graph of $g(x) = \dfrac{1}{f(x)}$?

A. (a,b)

B. (b,a)

C. $\left(a, \dfrac{1}{b}\right)$

D. $\left(\dfrac{1}{a}, b\right)$

E. $\left(\dfrac{1}{a}, \dfrac{1}{b}\right)$

Notice that this question has the restriction that $b \neq 0$. What would happen to the point on the reciprocal function if $b = 0$?

Solution

Answer: C

Justification: The reciprocal function $g(x) = \left(f(x)\right)^{-1}$ takes the reciprocal of the y-value for each point in f. Consider when $x = a$:

$$g(a) = \frac{1}{f(a)} \quad \text{when } x = a \qquad \left(a, f(a)\right) = (a, b)$$

$$g(a) = \frac{1}{b} \quad \text{since } f(a) = b \qquad \left(a, g(a)\right) = \left(a, \frac{1}{b}\right)$$

Notice that x-values are not changed. The point (a,b) transforms into the point $\left(a, \frac{1}{b}\right)$.

Do not get confused with the inverse of a function $g(x) = f^{-1}(x)$, which interchanges x and y-values. In general:

$$f^{-1}(x) \neq \left(f(x)\right)^{-1}$$

Summary

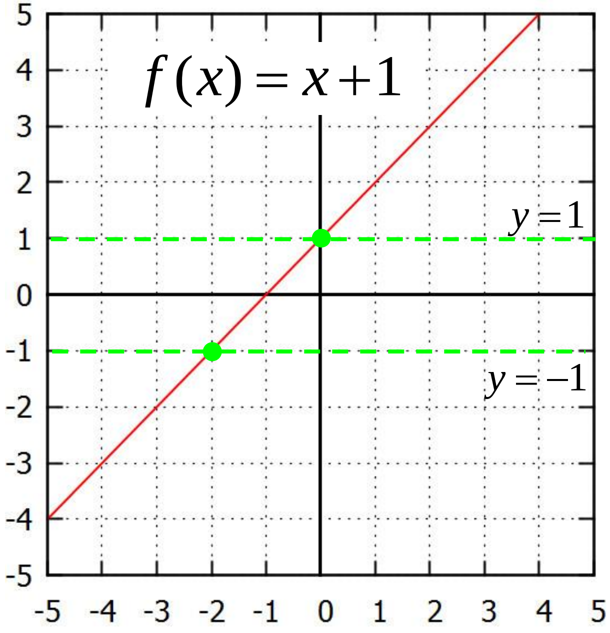
$f(x)$	$\left(f(x)\right)^{-1}$
$f(x) < -1$	$-1 < \left(f(x)\right)^{-1} < 0$
$f(x) = -1$	$\left(f(x)\right)^{-1} = -1$
$-1 < f(x) < 0$	$\left(f(x)\right)^{-1} < -1$
$f(x) = 0$	vertical asymptote
$0 < f(x) < 1$	$\left(f(x)\right)^{-1} > 1$
$f(x) = 1$	$\left(f(x)\right)^{-1} = 1$
$f(x) > 1$	$0 < \left(f(x)\right)^{-1} < 1$

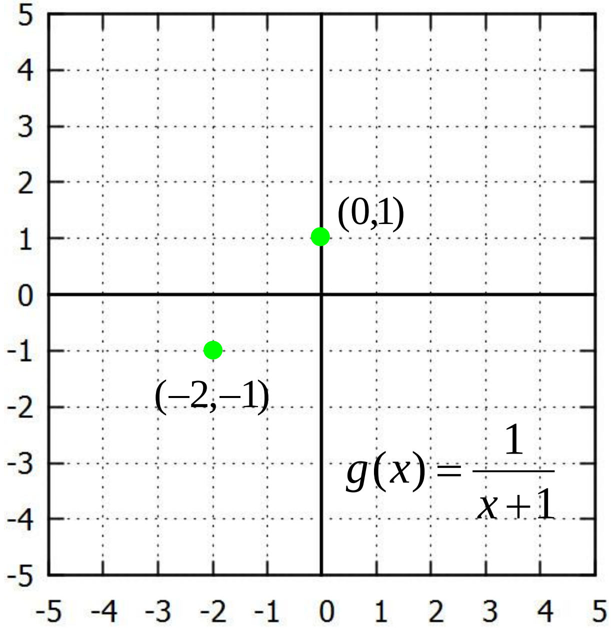
Strategy for Graphing I

Consider the function $f(x) = x + 1$ and its reciprocal function $g(x) = \dfrac{1}{x+1}$.

The following example outlines the steps to graph g :

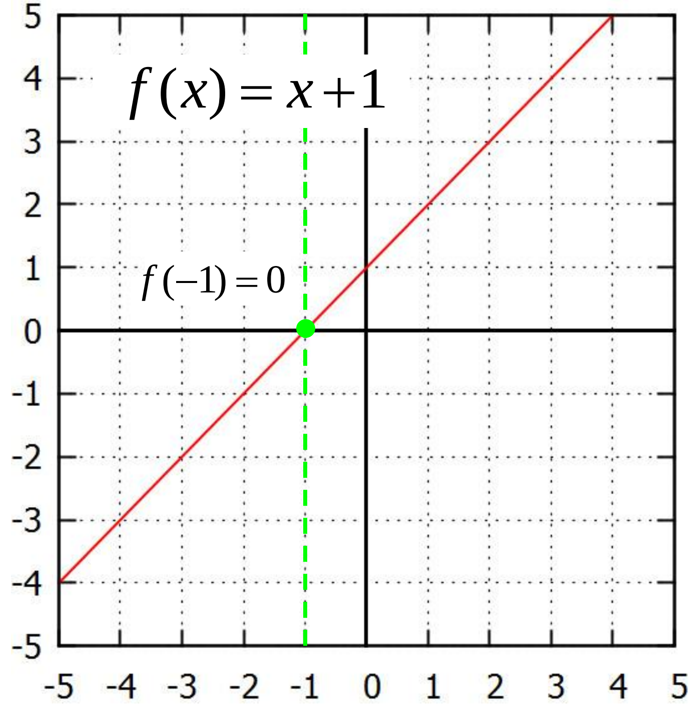
1. Identify the points where $f(x) = 1$ or $f(x) = -1$. These points exist on the reciprocal function. (See question 3)

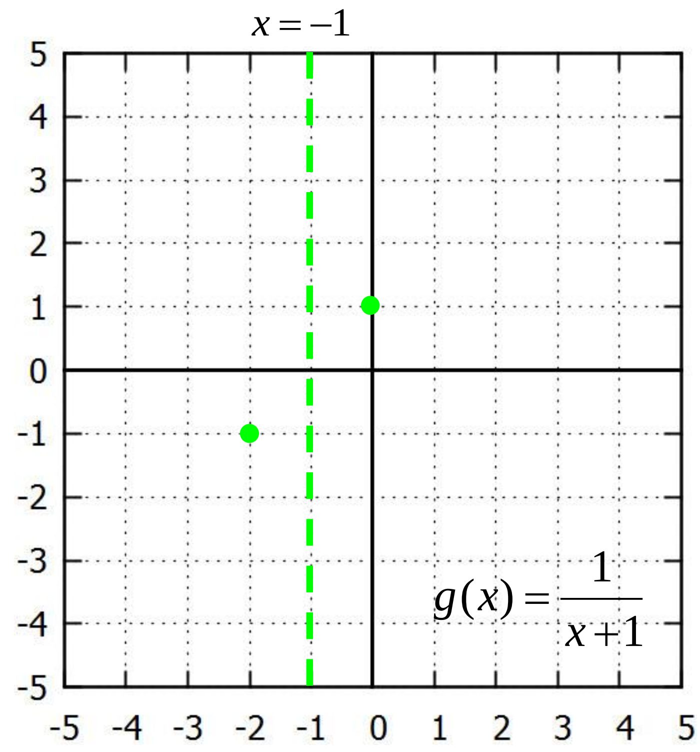




Strategy for Graphing II

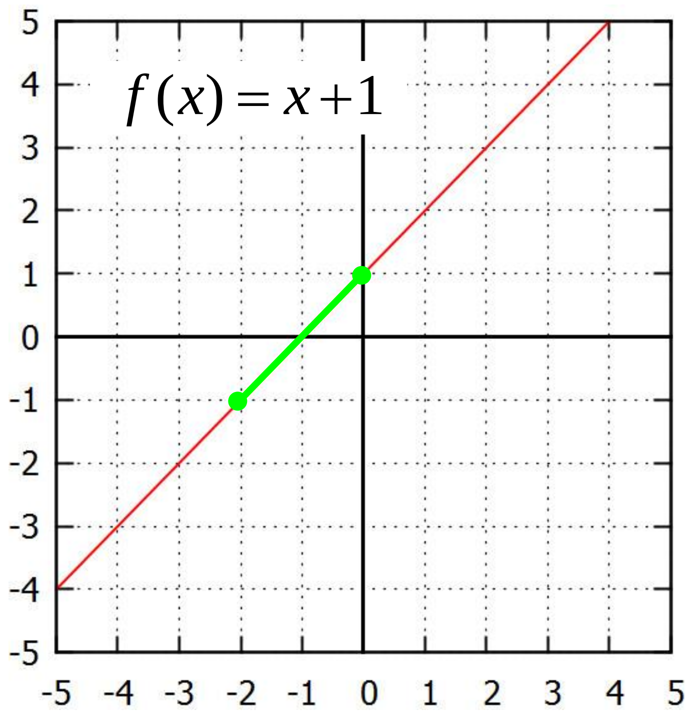
2. Identify the points where $f(x) = 0$. Draw a dotted vertical line through these points to show the locations of the vertical asymptotes. (Recall taking the reciprocal of 0 gives an undefined value.)

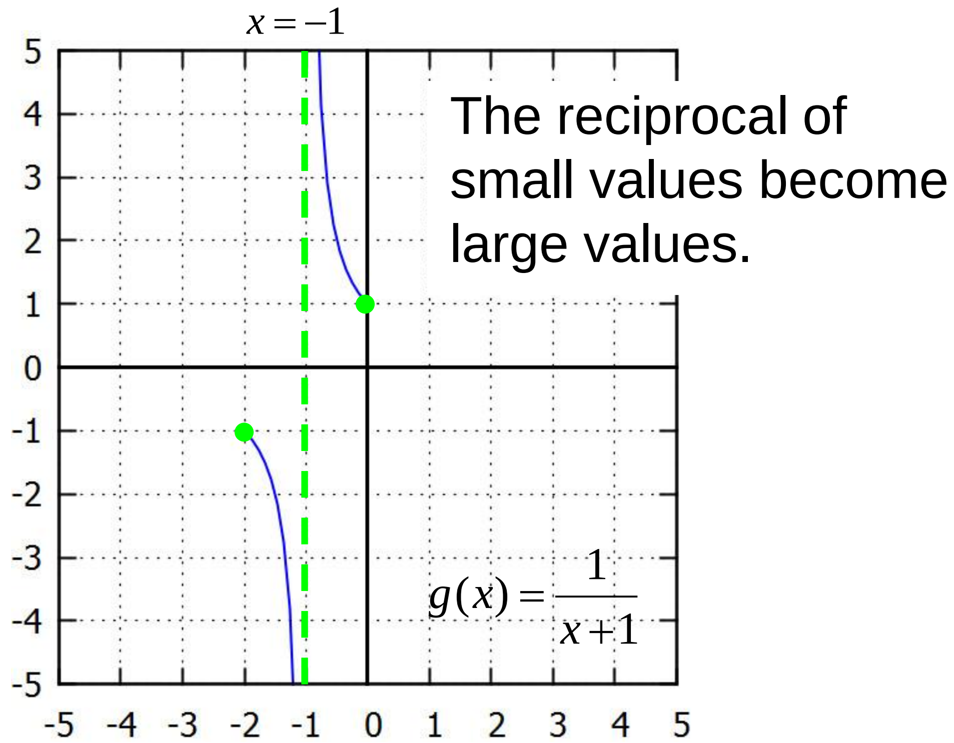




Strategy for Graphing III

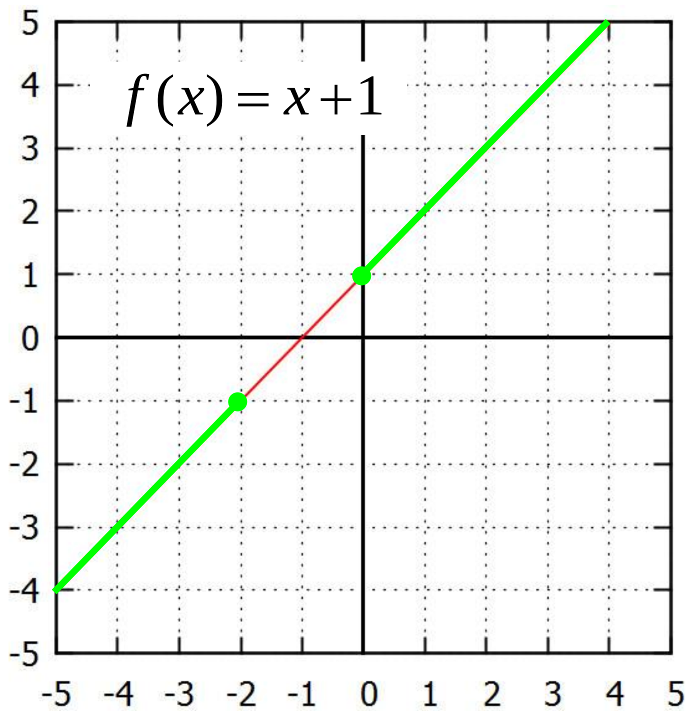
3. Consider when $0 < f(x) < 1$. Draw a curve from the top of the asymptote line to the point where $g(x) = 1$. When $-1 < f(x) < 0$, draw the curve from the bottom of the asymptote to $g(x) = -1$.

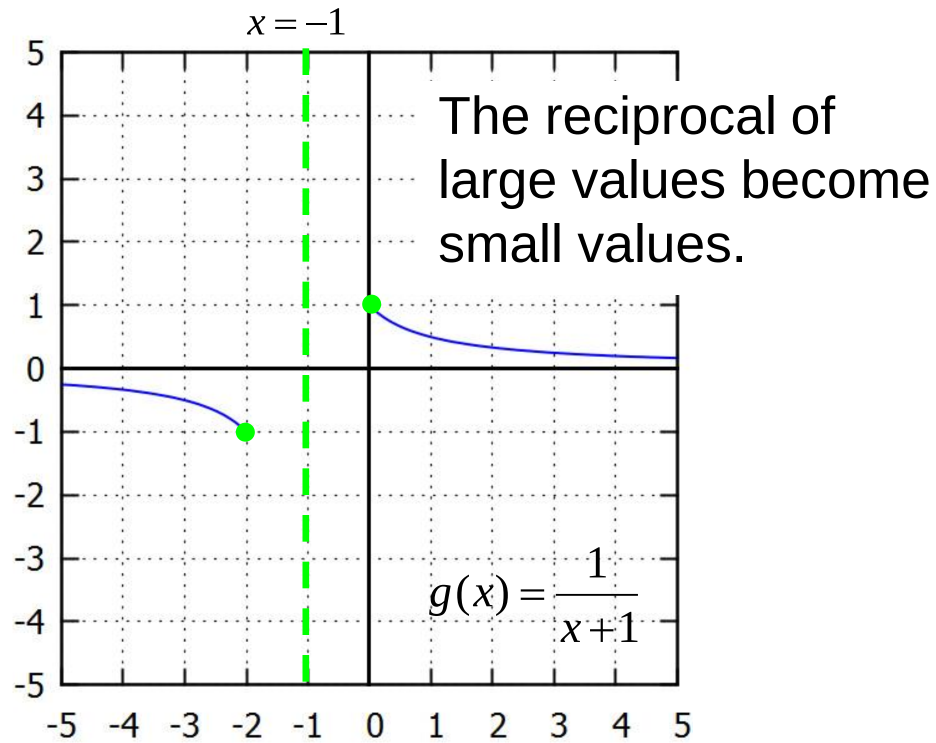




Strategy for Graphing IV

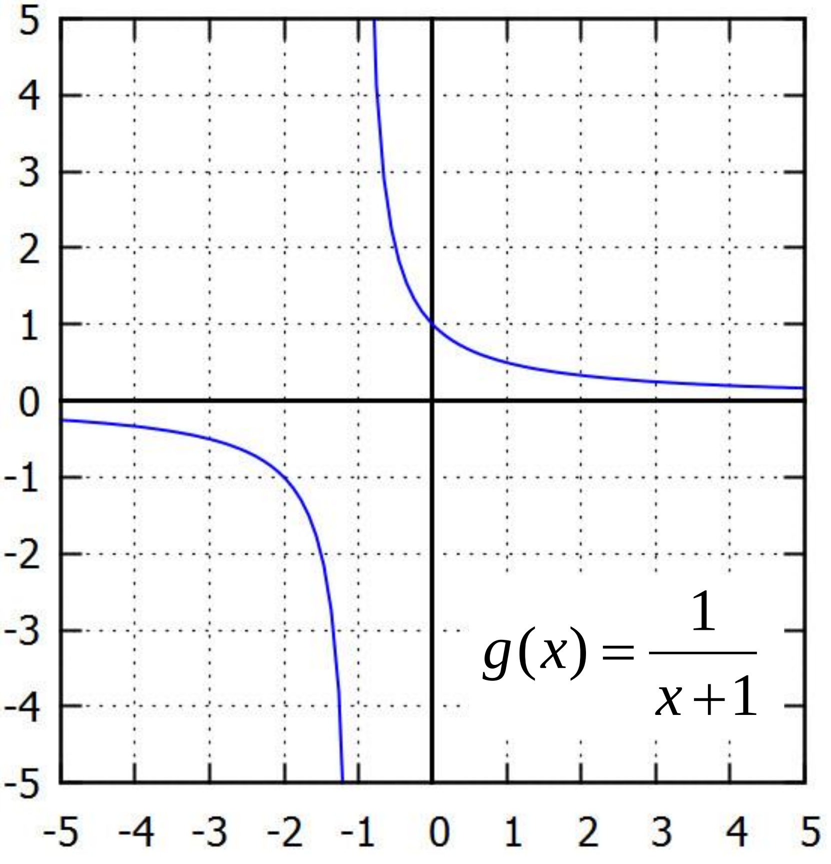
4. Consider where $f(x) > 1$. From the point $f(x) = 1$, draw a curve that approaches the x-axis from above. When $f(x) < 1$, the curve approaches the x-axis from below.

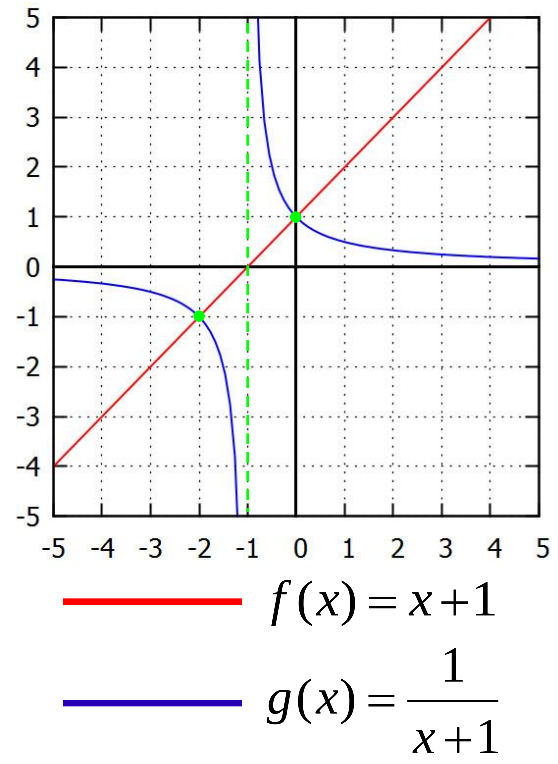




Strategy for Graphing V

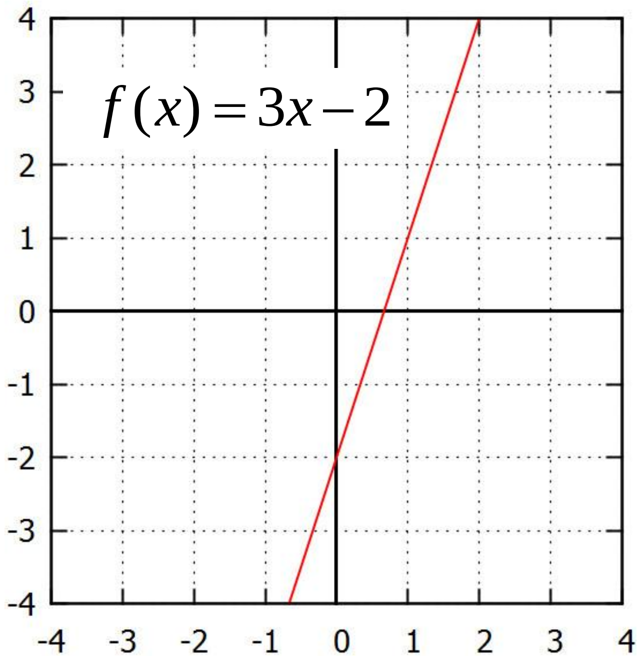
The final graph of $g(x) = \dfrac{1}{x+1}$ is shown below.



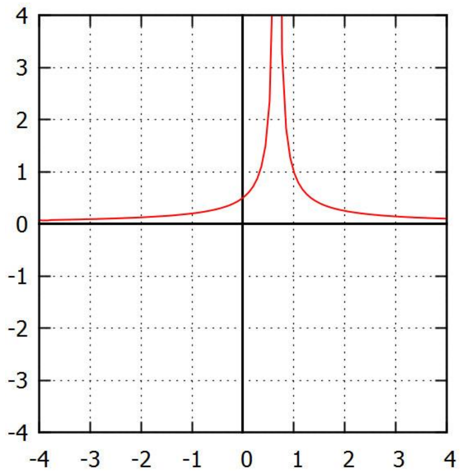


Reciprocal Functions VI

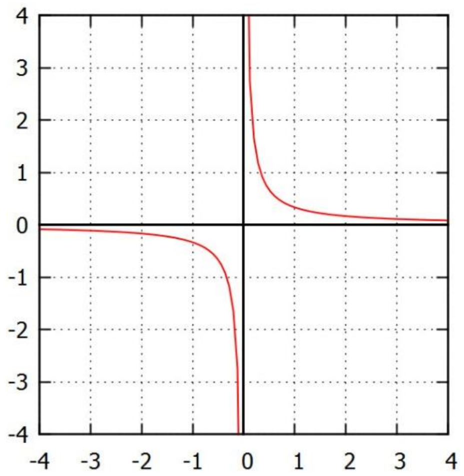
The graph of f is shown below. What is the correct graph of its reciprocal function, g?



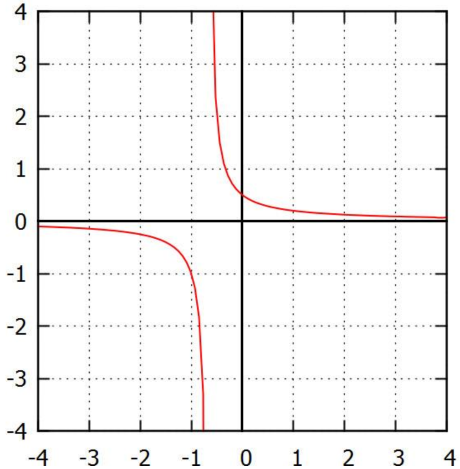
A.



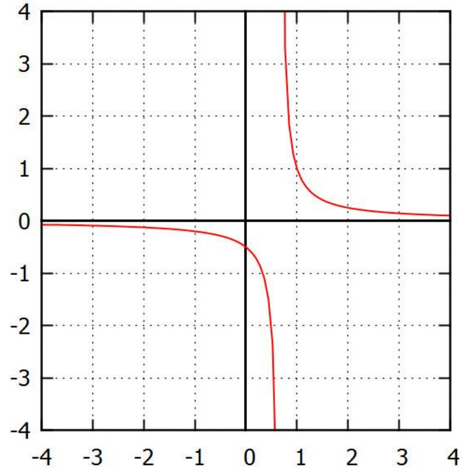
B.



C.



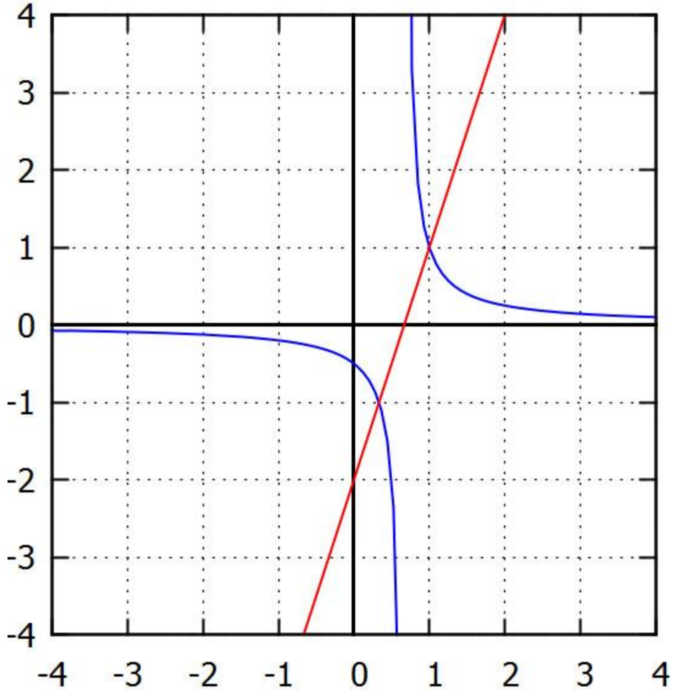
D.



Solution

Answer: D

Justification: Since $f(1)=1$, its reciprocal must also pass through the point (1,1). This eliminates answers B and C.



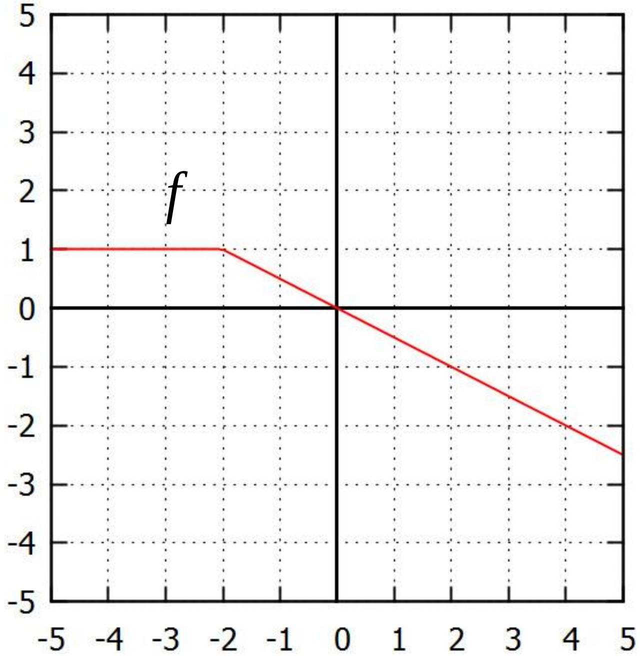
Additionally, f is positive for $x < \frac{2}{3}$ and negative for $x > \frac{2}{3}$.

This must also be true for the reciprocal function, so answer A must be eliminated because it is always positive. The only remaining answer is D.

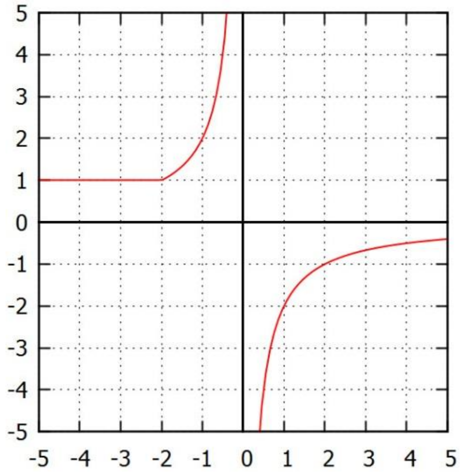
What other features can be used to identify the reciprocal function? What can we say about points $(1,1)$ and $(\frac{1}{3},-1)$?

Reciprocal Functions VII

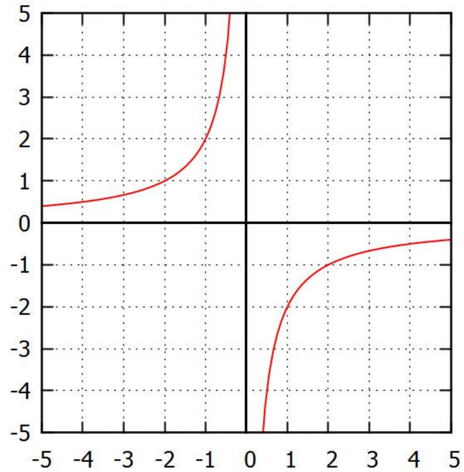
The graph of f is shown below. What is the correct graph of its reciprocal function, g?



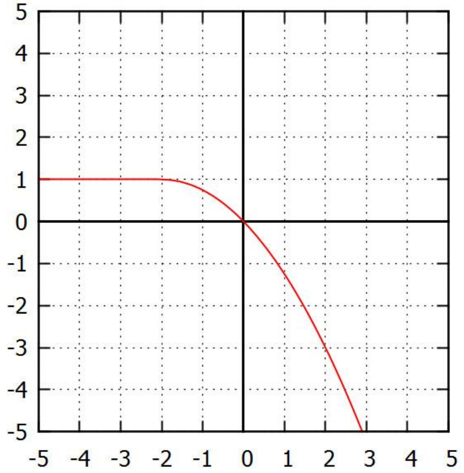
A.



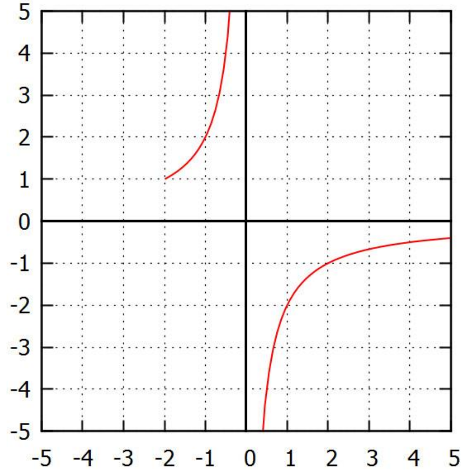
B.



C.



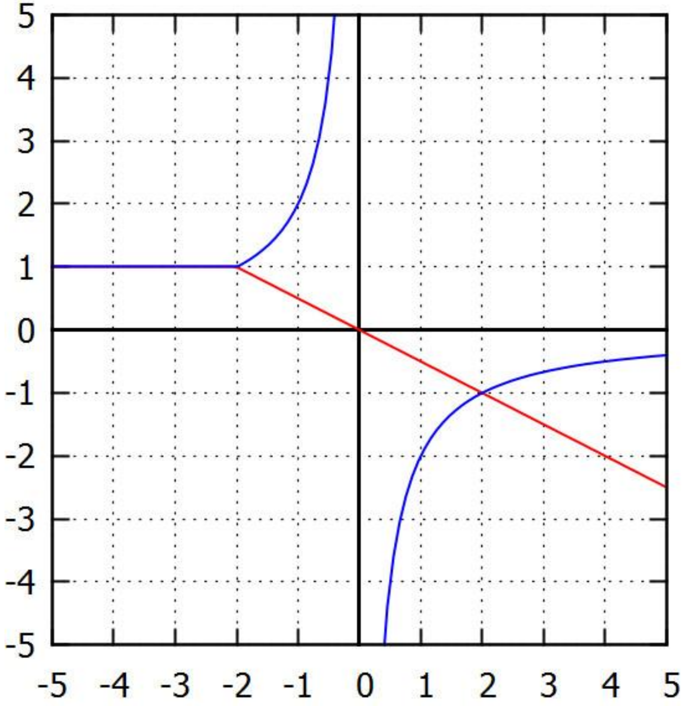
D.



Solution

Answer: A

Justification: When $x < -2$, $f(x) = 1$. The reciprocal function also has this horizontal line since the reciprocal of 1 is still 1. This eliminates answers B and D.



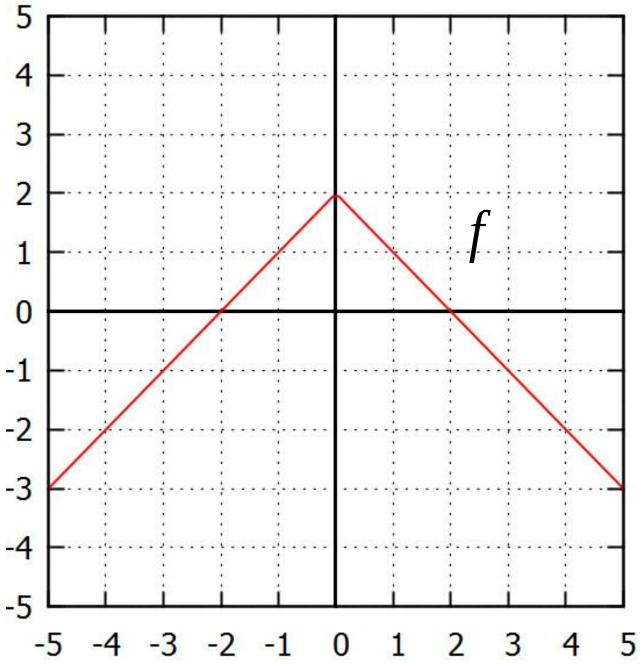
Since $f(0) = 0$, $g(x)$ must have an asymptote at $x = 0$. This only leaves function A.

$$f(x) = \begin{cases} 1, & x \le -2 \\ -\dfrac{x}{2}, & x > -2 \end{cases}$$

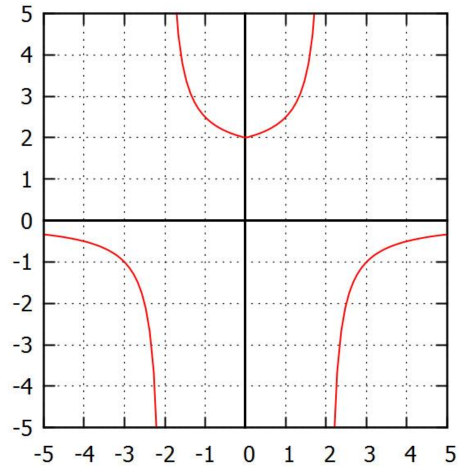
$$g(x) = \begin{cases} 1, & x \le -2 \\ -\dfrac{2}{x}, & x > -2 \end{cases}$$

Reciprocal Functions VIII

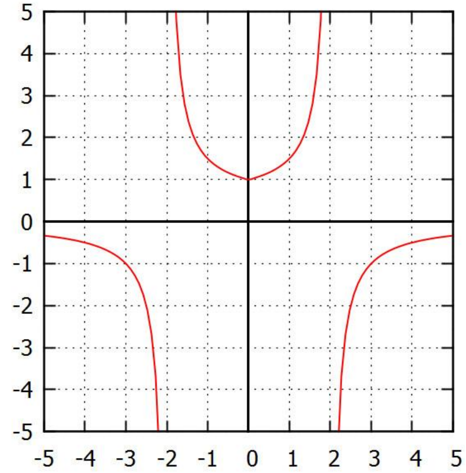
The graph of f is shown below. What is the correct graph of its reciprocal function, g?



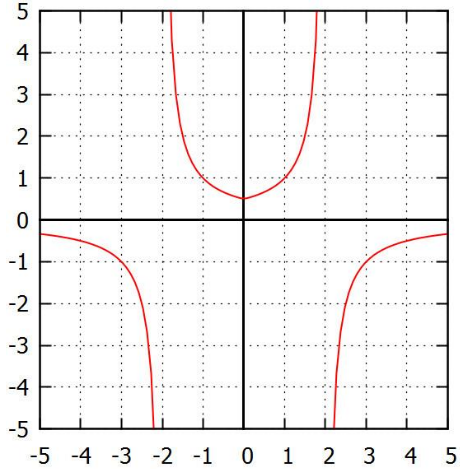
A.



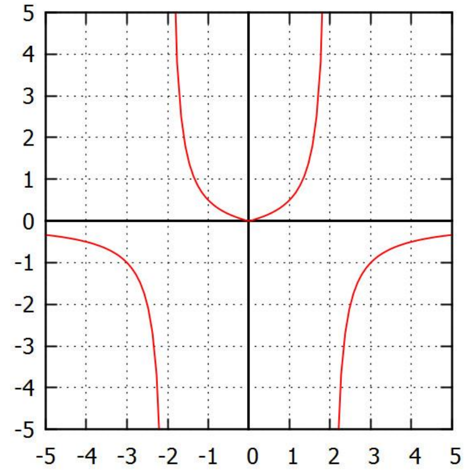
B.



C.



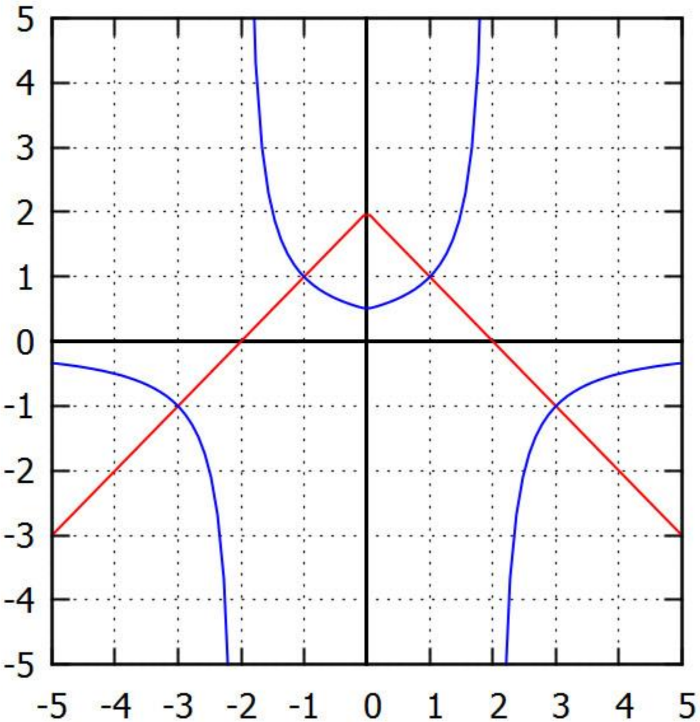
D.



Solution

Answer: C

Justification: Consider the point (0, 2) on f. This point will appear as $(0, \frac{1}{2})$ on g. Only function C passes through this point.



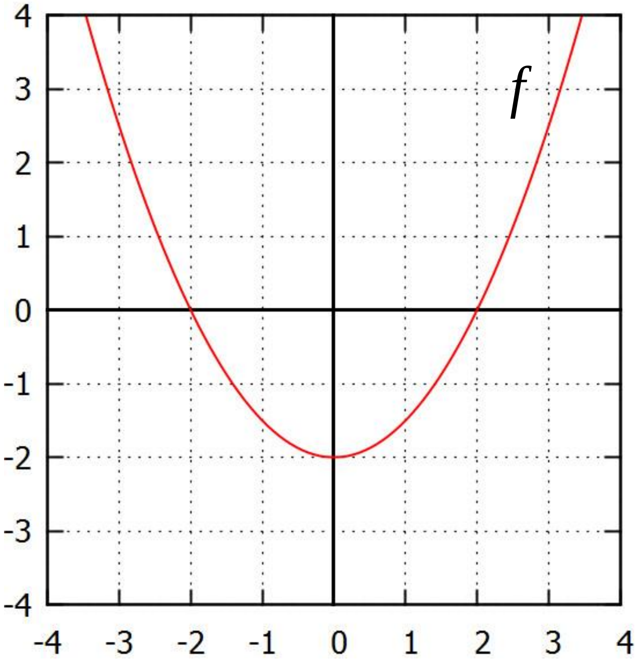
What other features can be used to identify the reciprocal function?

$$f(x) = \begin{cases} x+2, & x \leq 0 \\ -x+2, & x > 0 \end{cases}$$

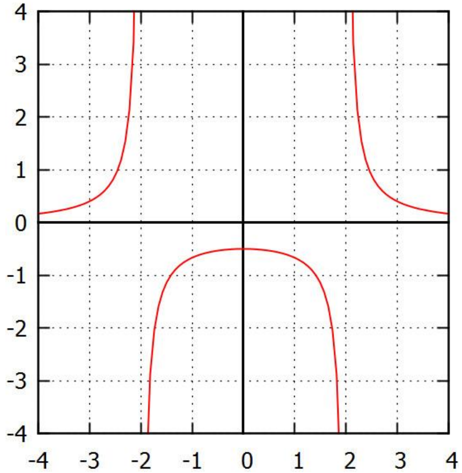
$$g(x) = \begin{cases} \dfrac{1}{x+2}, & x \leq 0 \\ \dfrac{1}{-x+2}, & x > 0 \end{cases}$$

Reciprocal Functions IX

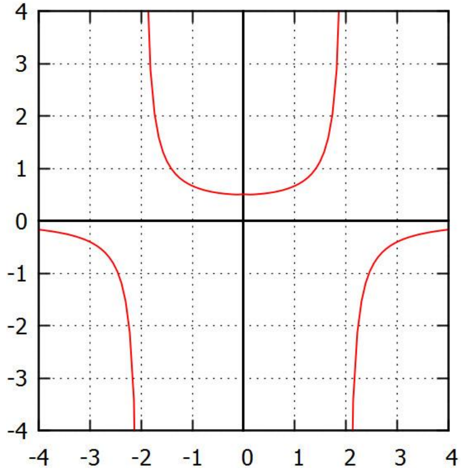
The graph of f is shown below. What is the correct graph of its reciprocal function, g?



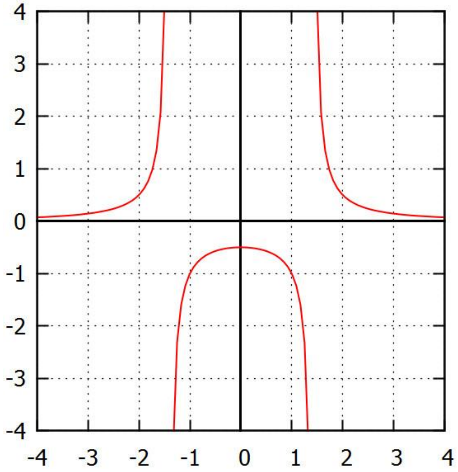
A.



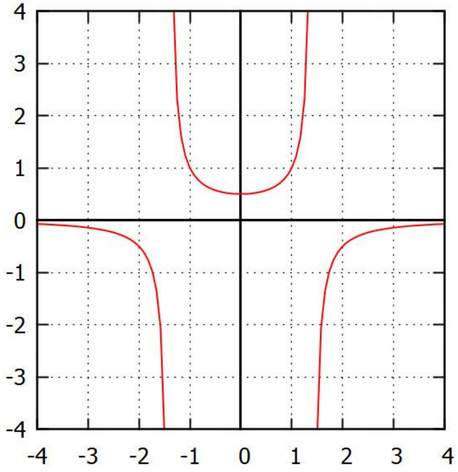
B.



C.



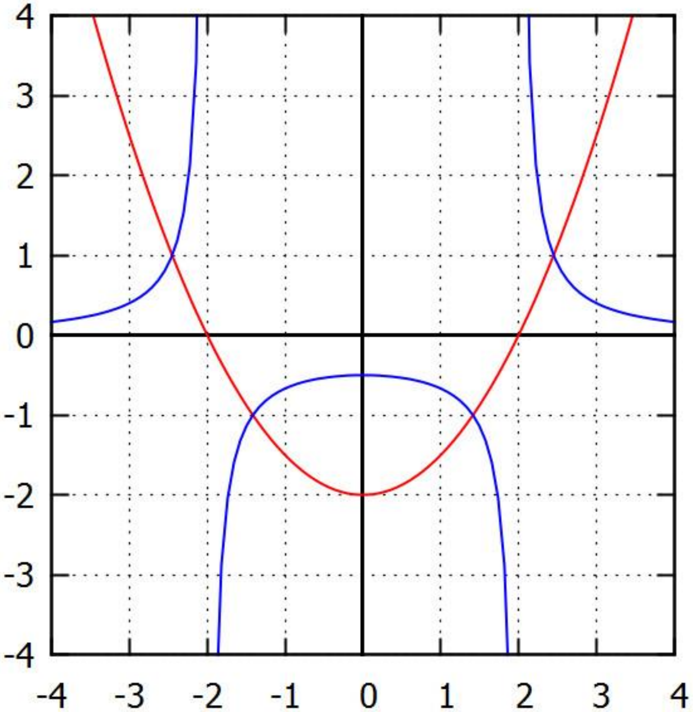
D.



Solution

Answer: A

Justification: The reciprocal function must be positive when the original function is positive, and negative when the original function is negative. This eliminates answers B and D.



The reciprocal function must have asymptotes at $x = \pm 2$ because the original function has zeroes at these values of x. Only graph A satisfies both of these conditions.

$$f(x) = 0.5x^2 - 2$$

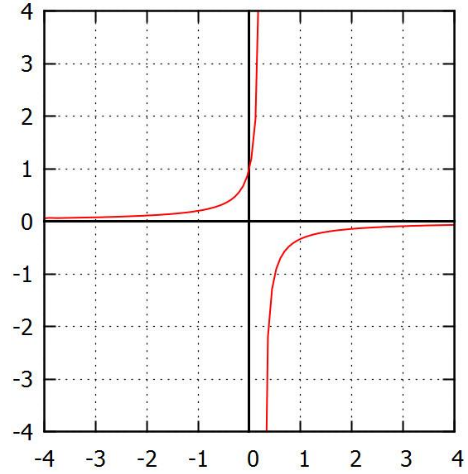
$$g(x) = \frac{1}{0.5x^2 - 2}$$

Reciprocal Functions X

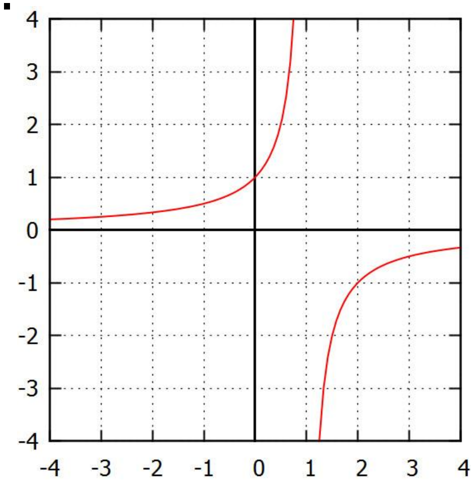
What is the graph of the reciprocal function

$$\left(f(x)\right)^{-1} = \frac{1}{-2x+1}?$$

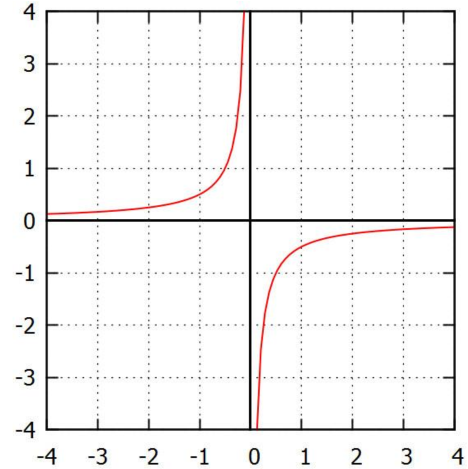
A.



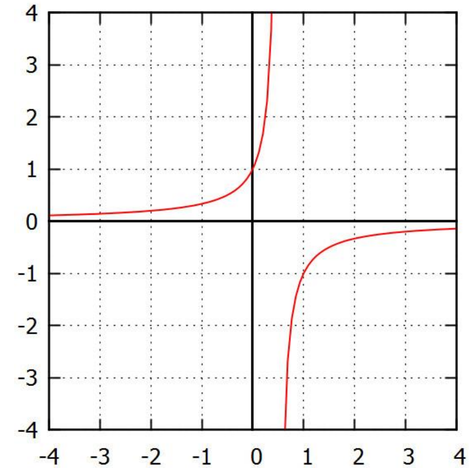
B.



C.



D.



Solution

Answer: D

Justification: Find where the original function $f(x) = -2x + 1$ crosses the lines $y = \pm 1, 0$:

$$-1 = -2x + 1$$

$$x = 1$$

$(f(x))^{-1}$ passes through the point (1, -1).

$$0 = -2x + 1$$

$$x = \frac{1}{2}$$

$(f(x))^{-1}$ has an asymptote at $x = 0.5$.

$$1 = -2x + 1$$

$$x = 0$$

$(f(x))^{-1}$ passes through the point (0, 1).

Only graph D satisfies these 3 conditions.

This question can also be solved by first graphing $f(x) = -2x + 1$, then determining the reciprocal graph.

Reciprocal Functions XI

The graph shown to the right is in the form:

$$\left(f(x)\right)^{-1} = \frac{1}{mx+b}$$

What are the values of *m* and *b*?

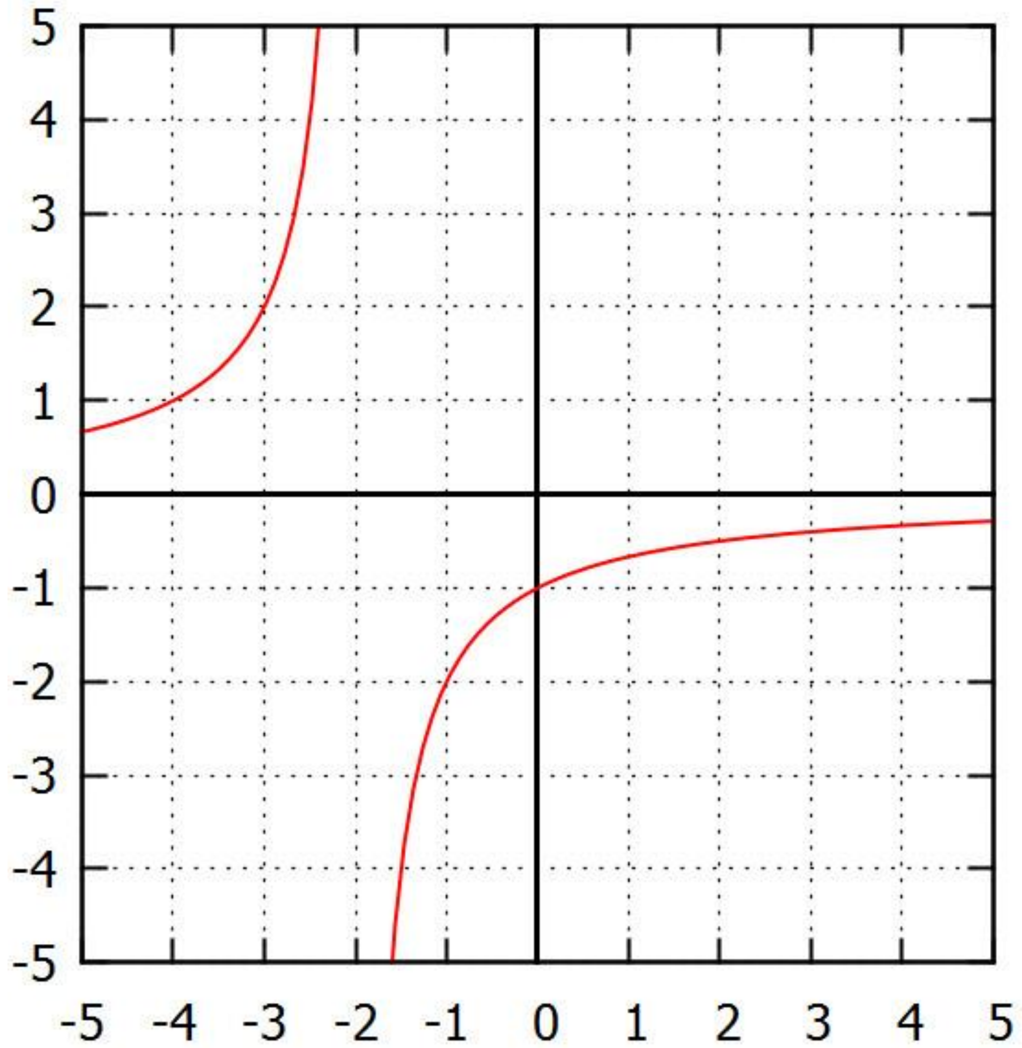
A. $m=-1, \quad b=-1$

B. $m=-2, \quad b=-1$

C. $m=-0.5, \quad b=-1$

D. $m=2, \quad b=1$

E. $m=0.5, \quad b=1$



Solution

Answer: C

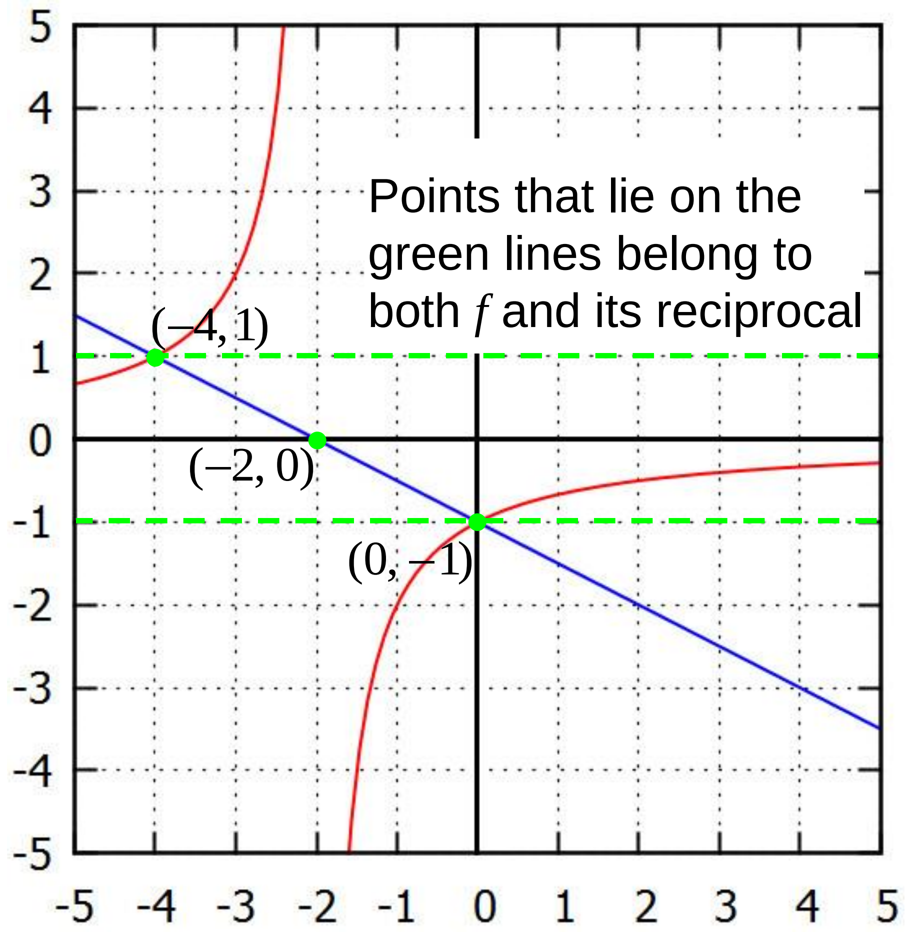
Justification: The function f is in the form $f(x) = mx + b$. We need to find a pair of points that lie on f to determine its equation. One good choice is (-4, 1) and (0, -1), since these points lie on both f and f^{-1}. Another choice for a point is (-2, 0), since there is an asymptote at x = -2.

Using any of the 3 points mentioned above, find the slope of the line:

$$m = \frac{(-1)-1}{0-(-4)} = -0.5$$

The y-intercept is at (0, -1), so

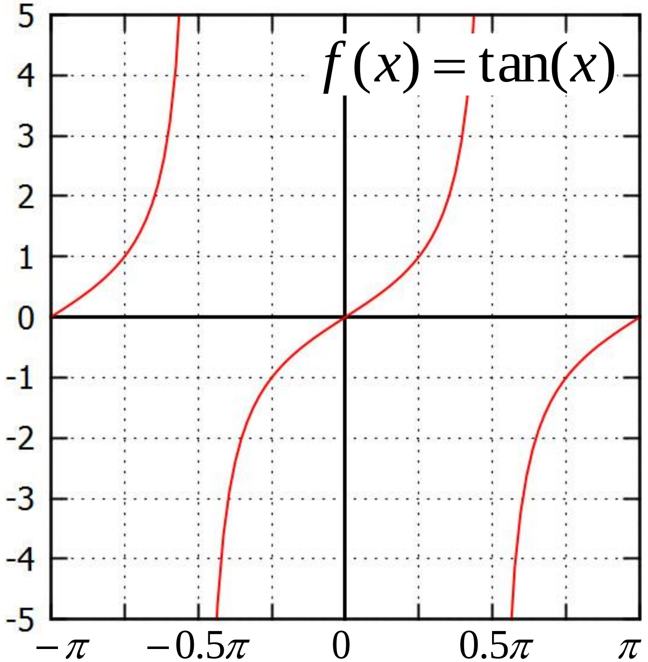
$$b = -1$$



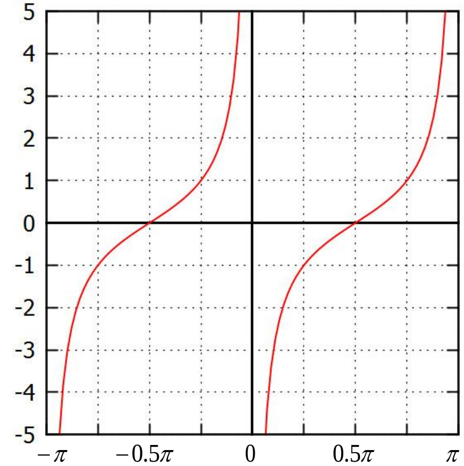
Reciprocal Functions XII

What is the graph of the cotangent function?

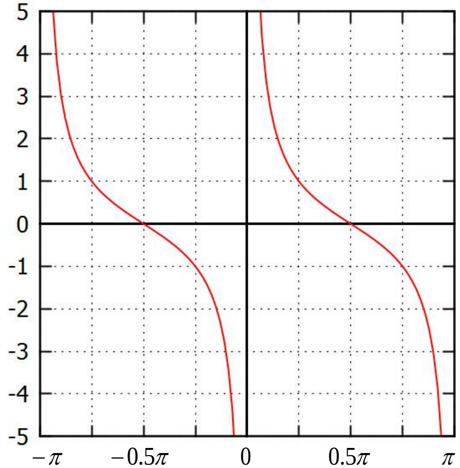
$$\left(f(x)\right)^{-1} = \cot(x) = \frac{1}{\tan(x)}$$



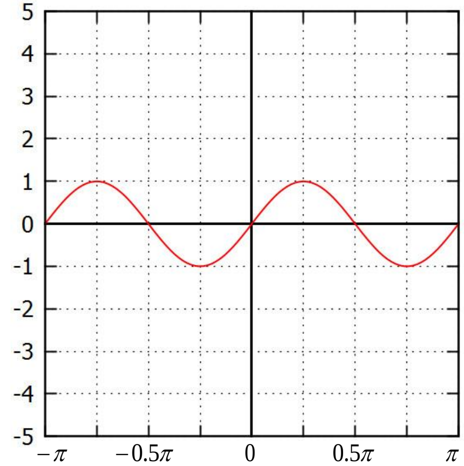
A.



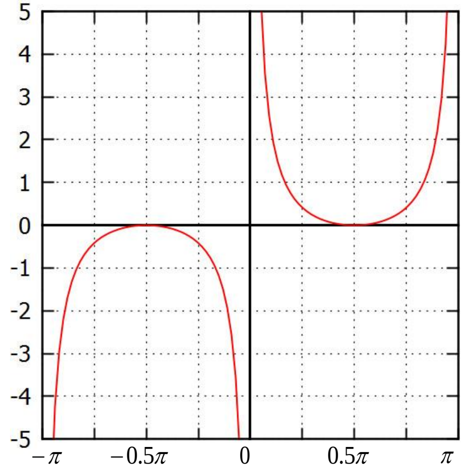
B.



C.



D.

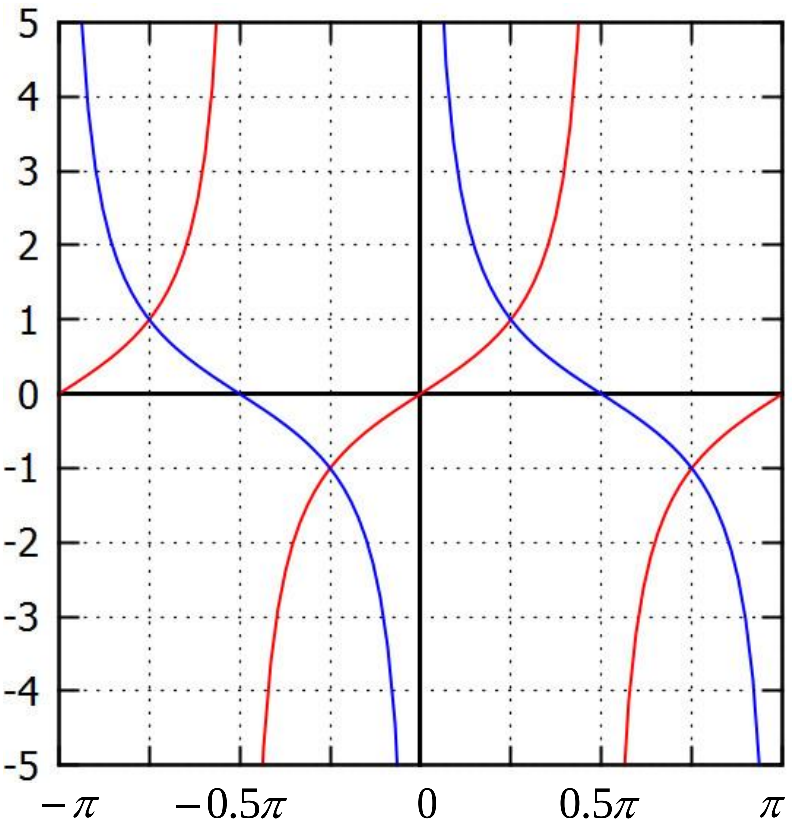


Solution

Answer: B

Justification: The points that lie on both the tangent and cotangent function are $\left(\pi/_4, 1\right)$ and $\left(-\pi/_4, -1\right)$.

The tangent function also has asymptotes at $x = \pi/_2 \pm k\pi$, where k is an integer. At these values of x, the cotangent function must return 0. The cotangent function has asymptotes at $x = k\pi$.



$$f(x) = \tan(x)$$

$$\left(f(x)\right)^{-1} = \cot(x) = \frac{1}{\tan(x)}$$