a place of mind

FACULTY OF EDUCATION

Department of
Curriculum and Pedagogy

Mathematics
Geometry: Menger Sponge

Science and Mathematics
Education Research Group

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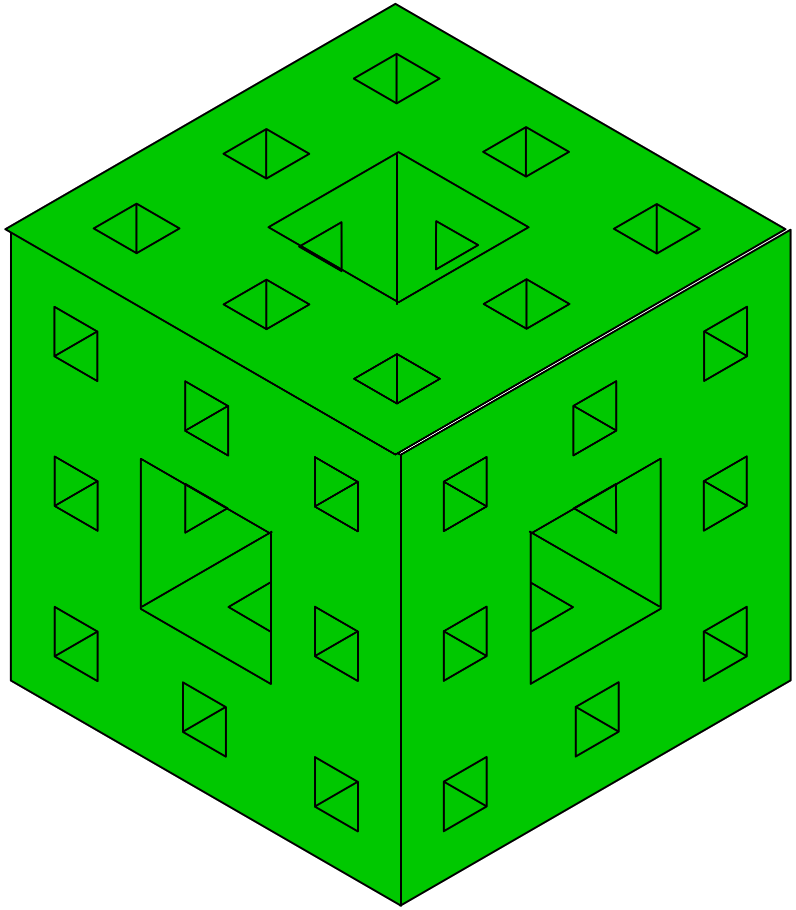
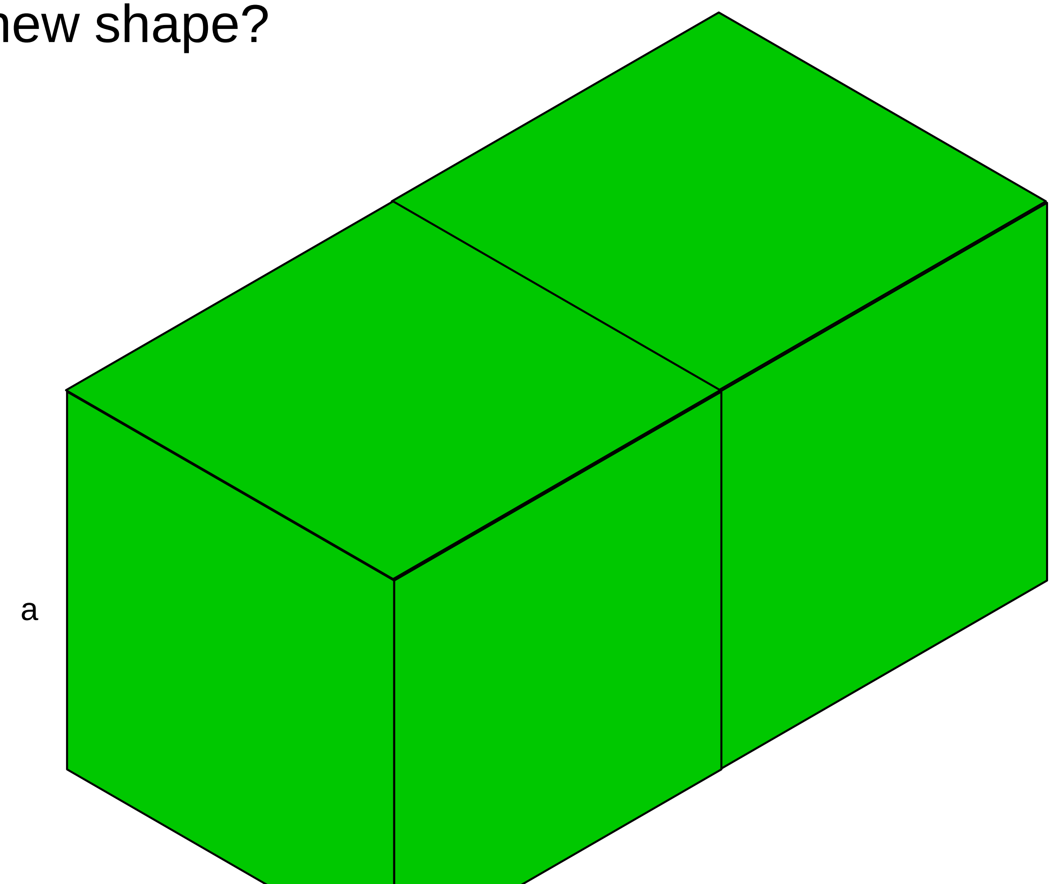
Menger Sponge



Menger Sponge I

Two identical cubes each with side length a are joined by a face. What is the surface area of the new shape?

A. $12a^2$
B. $11a^2$
C. $10a^2$
D. $6a^2$
E. No idea



Solution

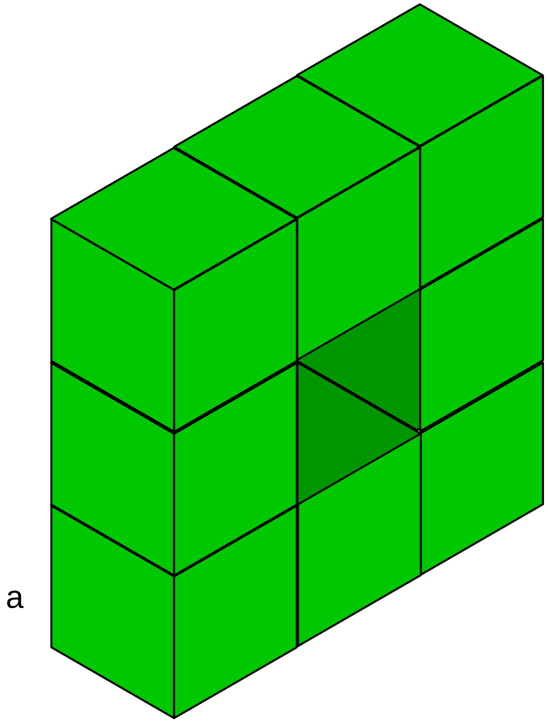
Answer: C

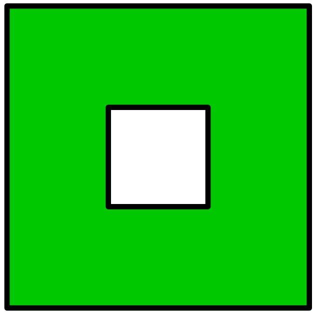
Justification: We know that a cube has a surface area of $6a^2$ because it has 6 sides. So 2 cubes alone will have a total surface area of $12a^2$. But, since 2 sides are joined when the cubes are joined, the total surface area will decrease by $2a^2$, so the final surface area is $10a^2$

Menger Sponge II

A. $48a^2$
B. $40a^2$
C. $32a^2$
D. $16a^2$
E. No idea

Eight identical cubes each with side length a are joined by their faces in a ring as shown below. What is the surface area of the ring?





side view

Solution

Answer: C

Justification: For 8 standalone cubes, the total surface area is $8 \times 6a^2 = 48a^2$. However, since 8 sides are touching in the diagram, an effective 16 sides have been removed from the shape, constituting $16a^2$ of the total $48a^2$. There is $32a^2$ of surface area remaining, which is our answer.

Menger Sponge III

A shape is formed by connecting 2 rings from the previous question with 4 cubes of side length a as shown. What is the surface area of the shape?

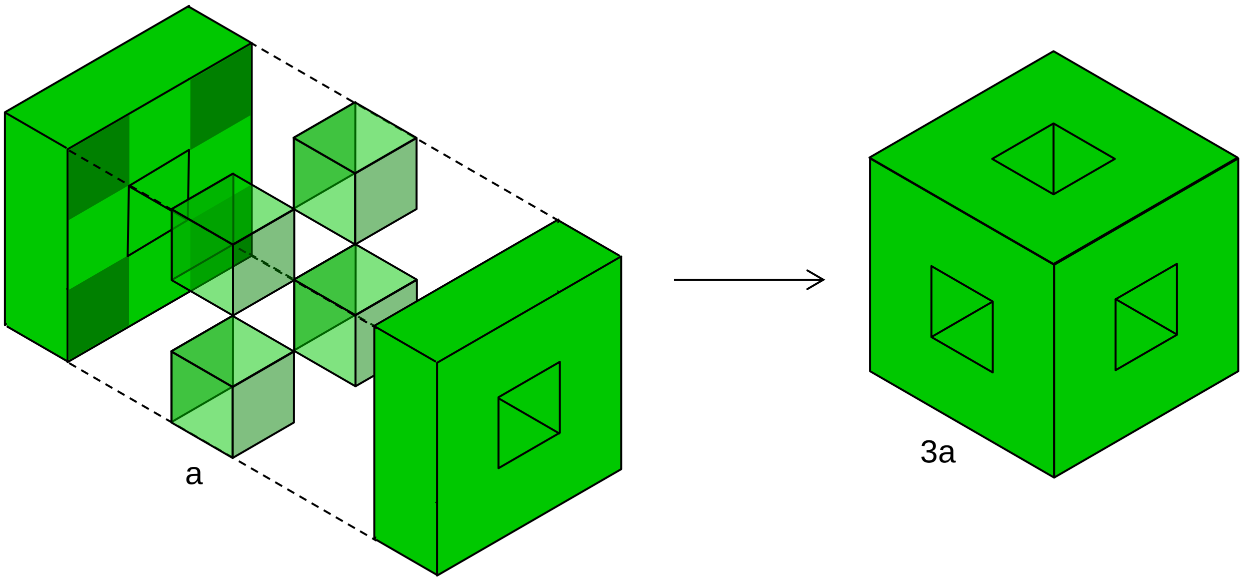
A. $88a^2$

B. $84a^2$

C. $80a^2$

D. $72a^2$

E. No idea



Solution

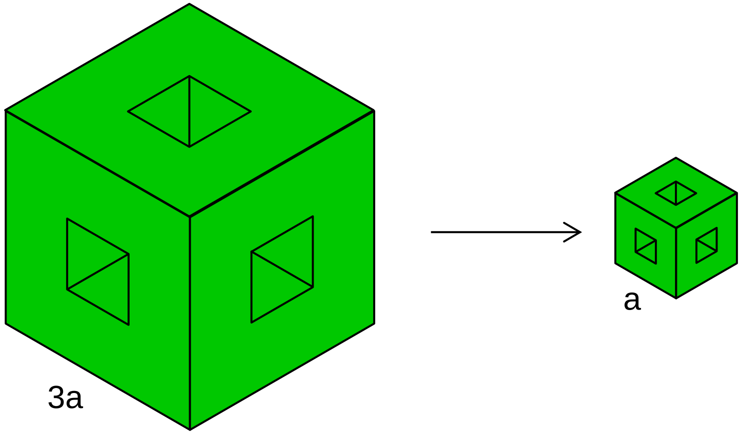
Answer: D

Justification: From question 2 we know that the surface area of the ring is $32a^2$. The standalone pieces of this shape is 2 rings, which have a total surface area of $2 \times 32a^2 = 64a^2$, and 4 cubes, which have a total surface area of $4 \times 6a^2 = 24a^2$, for a total separated area of $88a^2$. Since 8 squares are "shared" when the pieces are combined, 16 squares of area are essentially eliminated, removing a total of $16a^2$ of surface area. Therefore the final shape has an area of $72a^2$.

Menger Sponge IV

The shape from the previous question was reduced in size so that its side length is 3 times smaller. What is the surface area of the smaller shape?

A. $72a^2$
B. $18a^2$
C. $9a^2$
D. $8a^2$
E. No idea



Solution

Answer: D

Justification: The shape in the last question had a surface area of $72a^2$. Since each side length is decreased by a factor of 3, each square area is decreased by a factor of 3^2, or 9. $72a^2/9=8a^2$, which is the surface area of the final shape.

Menger Sponge V

A shape is formed by connecting twenty shapes from the previous question as shown below. What is the surface area of the shape produced? Dark green represents faces that are going to be covered.

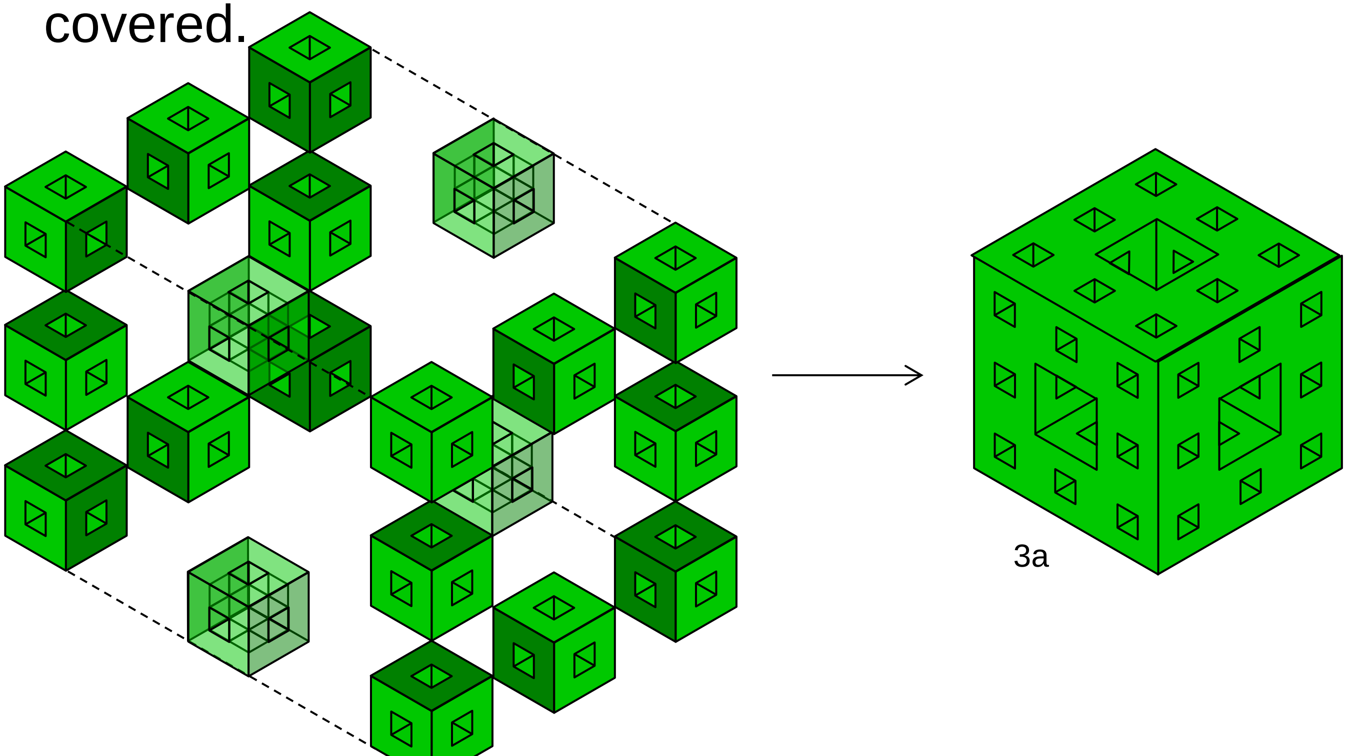
A. $(160-48\times8/9)a^2$

B. $(160-24\times8/9)a^2$

C. $(160-16\times8/9)a^2$

D. $160a^2$

E. No idea



Solution

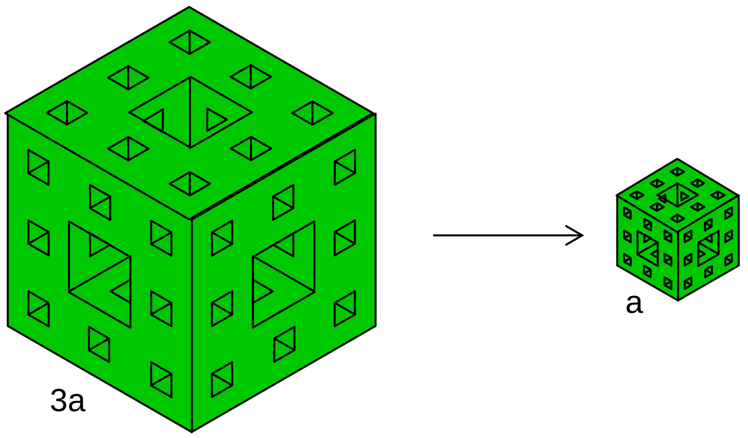
Answer: A

Justification: There are twenty of the shapes from the previous question, which we will call the 1st iteration shape. They contribute $20\times8a^2$ of surface area. As 48 sides are fused together in the process (16 for each "ring" and 16 for the center 4 pieces) and each face of the 1st iteration shape has $(8/9)a^2$ in area, the area of the shape produced will be $20\times8a^2-48(8/9)a^2$, which equals $(160-48\times8/9)a^2$.

Menger Sponge VI

Suppose the shape from the previous question has an area of A and was reduced in size so that its side length is 3 times smaller. What is the surface area of the smaller shape?

A. A/27
B. A/9
C. A/3
D. $8a^2$
E. No idea



Solution

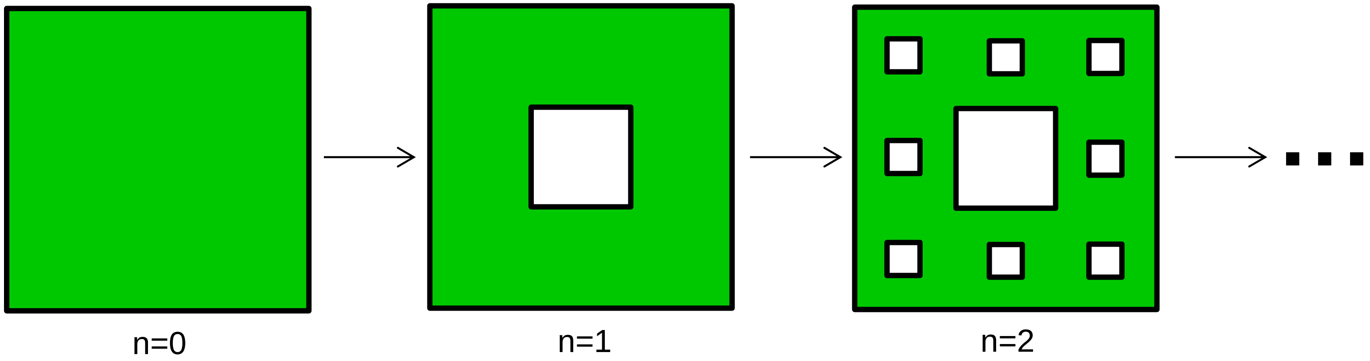
Answer: B

Justification: Each square area has its side length reduced by a factor of 3, so the area is reduced by a factor of $3^2=9$.

Menger Sponge VII

A. $(8/9)^n$
B. $(8/9)^{n-1}$
C. 8/9
D. $8/9^n$
E. No idea

The face of the shapes looks like the image below. Each time we add holes to the shape, its surface area is multiplied by 8, as 8 of them make the ring, and divided by 9, so the side length returns to the original shape. What would be the formula for the area in terms of n, if we start from an area of 1 (perfect square)?



Solution

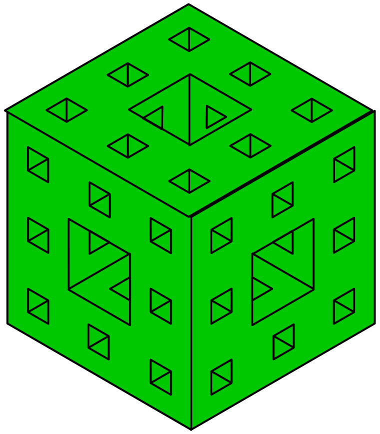
Answer: A

Justification: Every time n increases, the area is reduced by a factor of 8/9, which eliminates C and D. Since we start with an area of 1 at n=0, we must have $(8/9)^n$, as anything to the power of 0 is 1.

Menger Sponge VIII

The process of combining 20 shapes and then compressing the side length by a factor of 3 is called an iteration. That is why the shape in question 5 was referred to as the 1st iteration shape. Which of the choices is the formula that summarizes the change in surface area after an iteration? (A_{n-1} is the surface area of the previous shape)

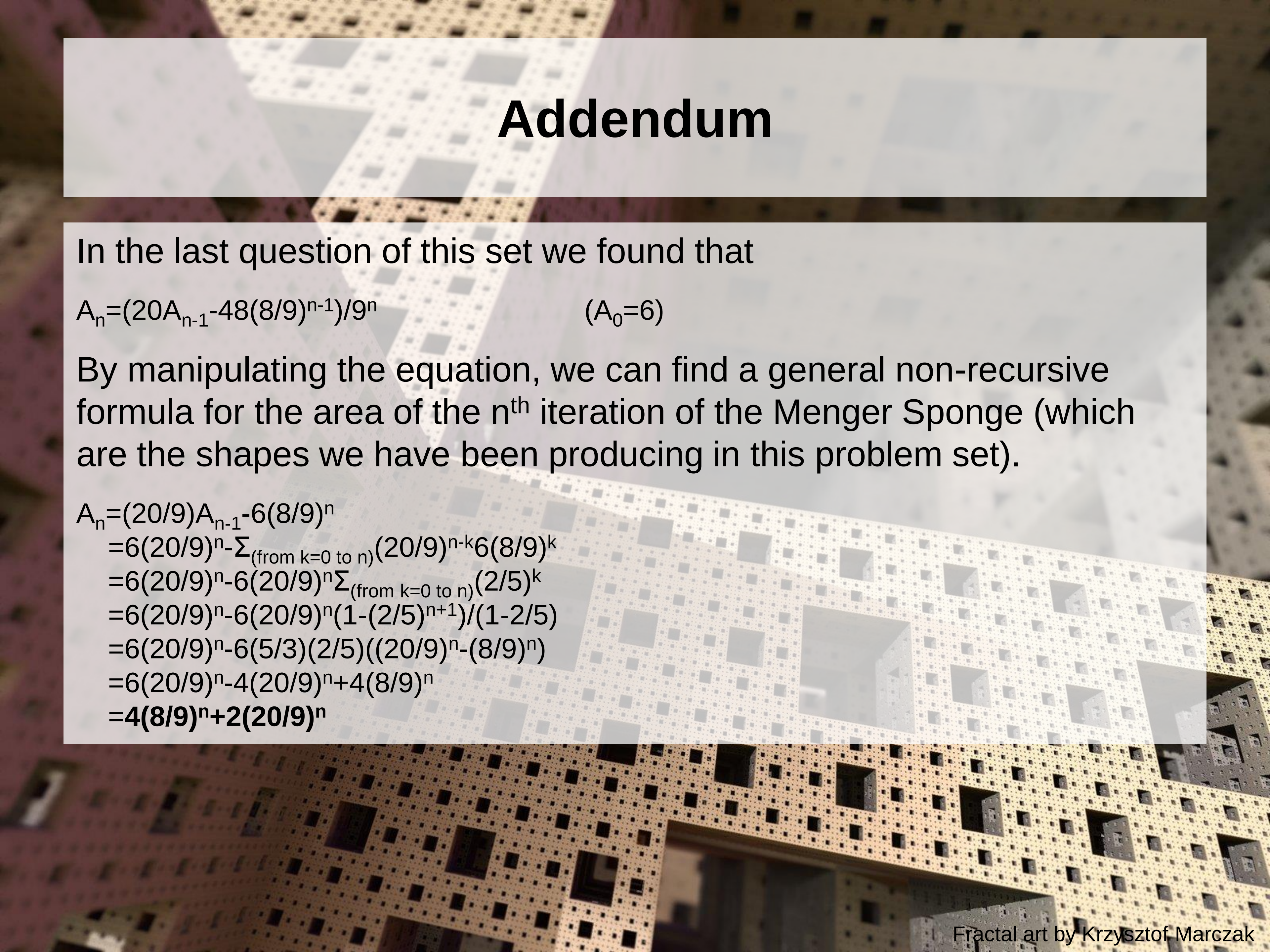
A. $(20A_{n-1}-48(8/9))/9$
B. $(20A_{n-1}-24(8/9)^{n-1})/9$
C. $(20A_{n-1}-48(8/9)^{n-1})/9$
D. $(20A_{n-1}-16(8/9))/9$
E. No idea



Solution

Answer: C

Justification: We have 20 shapes of the previous iteration, which separated have an area of $20A_{n-1}$. The 8 locations which are eliminated when the shapes are combined constitute 16 areas multiplied by $(8/9)^{n-1}$, the area of the previous shape's face. Our combined object now has $20A_{n-1}-48(8/9)^{n-1}$. After scaling the side length down to a third, we have A_n, the area of the current iteration, is equal to $(20A_{n-1}-48(8/9)^{n-1})/9$



Addendum

In the last question of this set we found that

$A_n=(20A_{n-1}-48(8/9)^{n-1})/9^n$ $(A_0=6)$

By manipulating the equation, we can find a general non-recursive formula for the area of the n^{th} iteration of the Menger Sponge (which are the shapes we have been producing in this problem set).

$A_n=(20/9)A_{n-1}-6(8/9)^n$

$=6(20/9)^n-\Sigma_{(\text{from } k=0 \text{ to } n)}(20/9)^{n-k}6(8/9)^k$

$=6(20/9)^n-6(20/9)^n\Sigma_{(\text{from } k=0 \text{ to } n)}(2/5)^k$

$=6(20/9)^n-6(20/9)^n(1-(2/5)^{n+1})/(1-2/5)$

$=6(20/9)^n-6(5/3)(2/5)((20/9)^n-(8/9)^n)$

$=6(20/9)^n-4(20/9)^n+4(8/9)^n$

$=\mathbf{4(8/9)^n+2(20/9)^n}$

Fractal art by Krzysztof Marczak

Addendum

This problem set was intended as an exercise to calculate total surface area of combinations of shapes by eliminating overlapping areas. There is a simpler way of deducing the area of a Menger Sponge:

Start with the area of the previous iteration (A_{n-1})
Remove the middle of all the faces ($(8/9)A_{n-1}$)
Add the internal surface area, which is 20^{n-1}(smaller cubes)×6(holes)×4(sides)×($1/9^n$)(scaling down) ($A_n=(8/9)A_{n-1}+(4\times6\times20^{n-1})/9^n$)

By manipulating the recurrence relation, we can see that we have the same general solution as before:

$$\begin{aligned}
A_n &= (8/9)A_{n-1}+(6/5)(20/9)^n \\
&= 6(8/9)^n+\Sigma_{(\text{from } k=0 \text{ to } n)}(8/9)^{n-k}(6/5)(20/9)^k \\
&= 6(8/9)^n+(6/5)(8/9)^n\Sigma_{(\text{from } k=0 \text{ to } n)}(5/2)^k \\
&= 6(8/9)^n+(6/5)(8/9)^n((5/2)^{n+1}-1)/(5/2-1) \\
&= 6(8/9)^n+(6/5)(2/3)(5/2)((20/9)^n-(8/9)^n) \\
&= 6(8/9)^n+2(20/9)^n-2(8/9)^n \\
&= \mathbf{4(8/9)^n+2(20/9)^n}
\end{aligned}$$