a place of mind

# Mathematics <br> Number: Logarithms 

## Science and Mathematics Education Research Group

## Logarithms



## Introduction to Logarithms

Find the value of $x$ where $2^{x}=8$.
A. $x=2$
B. $x=3$
C. $x=4$
D. $x=8$
E. $x=16$

Exponent law:
$a^{x}=a^{y} \quad$ if and only if $x=y$

## Solution

Answer: B
Justification: Try writing the right hand side of the equation as a power of two.

$$
8=2^{3}
$$

Then use the exponent law: $a^{x}=a^{y}$ if and only if $x=y$

$$
\begin{aligned}
2^{x} & =8 \\
2^{x} & =2^{3} \\
x & =3
\end{aligned}
$$

## Introduction to Logarithms II

Find the value of $x$ where $2^{x}=18$.
A. $x=3$
B. $x=4$
C. $x=5$
D. $x=6$
E. None of the above

## Solution

Answer: E
Justification: We cannot write 18 as a power of 2 . This means we cannot solve this question the same way as the previous one.

| $\boldsymbol{x}$ | $\mathbf{2}^{\boldsymbol{x}}$ |  |
| :---: | :---: | :--- |
| 3 | 8 |  |
| 4 | 16 | Although we can't find the exact answer, we |
| 5 | 32 | can guess the answer is between 4 and 5. |
| 6 | 64 |  |

The answer cannot be written as an integer, so the correct answer is E, "None of the above."

In order to answer this question, we will have to learn about logarithms.

## Introduction to Logarithms III

If $y=a^{x}$ then $\log _{a} y=x$. Notice how the logarithm $\left(\log _{a} y\right)$ returns the exponent of $a^{x}$.

Using the above definition, what is the value of $\log _{2} 8$ ?
A. $\log _{2} 8=2$
B. $\log _{2} 8=3$
C. $\log _{2} 8=4$
D. $\log _{2} 8=5$
E. None of the above

## Solution

## Answer: B

Justification: This question is exactly the same as question 1. We must find the value of $x$ where $2^{x}=8$.

In question 1, we were asked to find the exponent $x$ such that $2^{x}=8$. This is exactly the same as finding the value of $\log _{2} 8$.

$$
\log _{2} 8=3 \quad \text { since } \quad 2^{3}=8
$$

In general, we can convert between logs and exponents as follows:


A helpful reminder:
equals $y$


## Introduction to Logarithms IV

Find the value of $x$ where $2^{x}=18$. Express the answer using logarithms.

Hint: Try expressing the question in terms of logarithms.
A. $x=\log _{2} 18$
B. $x=\log _{18} 2$
C. $x=\log _{2} x$
D. $x=\log _{2} y$
E. None of the above

$$
\text { If } y=a^{x} \text { then } \log _{a} y=x \text {. }
$$

Press for hint

## Solution

## Answer: A

Justification: Recall how equations in the form $y=a^{x}$, can be rewritten with logarithms, and vice-versa:

$$
\log _{a} \begin{gathered}
y=a^{x} \\
y \\
\downarrow
\end{gathered}
$$

$$
\begin{aligned}
2^{x} & =18 \\
x & =\log _{2} 18
\end{aligned}
$$

Notice how applying this to $2^{x}=18$ allows us to solve for $x$, which we were not able to do without logarithms:
This is the exact answer to the equation above, so $2^{\log _{2} 18}=18$. We will see later that this result can be generalized to a property of logarithms: $b^{\log _{b} a}=a$

## Logarithm Laws

The following is a summary of important logarithm properties:

$$
\begin{aligned}
& \log a b=\log a+\log b \\
& \log \frac{a}{b}=\log a-\log b
\end{aligned}
$$

$$
\log a^{n}=n \log a
$$

$$
\log _{b} a=\frac{\log _{c} a}{\log _{c} b}
$$

Other useful properties include:

$$
\begin{array}{lc}
\log _{b} 1=0 & \log _{b} \frac{1}{a}=-\log _{b} a \\
\log _{b} b=1 & b^{\log _{b} a}=a \\
\log _{b} x=\log _{b} y & \text { if and only if } \quad x=y
\end{array}
$$

Note: When you see $\log x$, it is assumed that the base of the logarithm is 10.

$$
\log x=\log _{10} x
$$

When you see $\operatorname{In} x$, the base is assumed to be $e \approx 2.72$.

$$
\ln x=\log _{e} x
$$

## Properties of Logarithms I

What is $\log _{7} a+\log _{7} b-\log _{7} c$ written as a single logarithm?
A. $\log _{7}(a+b-c)$
B. $\log _{7}(a b c)$
C. $\frac{\log _{7}(a b)}{\log _{7} c}$
D. $\log _{7}\left(\frac{a b}{c}\right)$
E. $\quad \log _{7}\left(\frac{c}{a b}\right)$

## Solution

## Answer: D

Justification: We will use the two following two properties to simplify the expression:

$$
\text { 1. } \log a b=\log a+\log b \quad \text { 2. } \log \left(\frac{a}{b}\right)=\log a-\log b
$$

First combine the terms $\log _{7} a+\log _{7} b=\log _{7} a b$ using the first property 1 shown above:

$$
\log _{7} a+\log _{7} b-\log _{7} c=\log _{7} a b-\log _{7} c
$$

Next combine the term that is subtracted using property 2 :

$$
\log _{7} a b-\log _{7} c=\log _{7}\left(\frac{a b}{c}\right)
$$

Does order matter? Would simplifying the subtracted term first change the answer?

## Properties of Logarithms II

What is the value of $\log _{10} 100^{8}$.
A. $100^{8}$
B. $10^{16}$
C. $10^{8}$
D. 16
E. 8

## Solution

## Answer: D

Justification: Use exponent and logarithm laws to simplify the problem. We can start by pulling the exponent the outside of the logarithm:

$$
\begin{aligned}
\log _{10} 100^{8} & =8 \log _{10} 100 & & \text { since }
\end{aligned} \quad \begin{array}{ll} 
& \log _{b} a^{n}=n \log _{b} a \\
& =8 \cdot 2
\end{array}
$$

The solution tells us that $10^{16}=100^{8}$.

## Alternative Solution

## Answer: D

Justification: Since the logarithm is in base 10, another strategy is to write all terms with a base of 10 .

$$
\begin{aligned}
& \log _{10} 100^{8}=\log _{10}\left(10^{2}\right)^{8} \\
& =\log _{10} 10^{16} \quad \text { since } \quad\left(a^{x}\right)^{y}=a^{x y} \\
& =16 \log _{10} 10 \quad \text { since } \quad \log _{b} a^{n}=n \log _{b} a \\
& =16 \\
& \text { since } \log _{b} b=1
\end{aligned}
$$

## Properties of Logarithms III

Which of the following is not equivalent to the following equation, where $b>0, b \neq 1$ ?

$$
\log _{b} b a^{b}=c
$$

A. $b^{c}=b a^{b}$
B. $\log _{b} b+b \cdot \log _{b} a=c$
C. $\log _{b} a^{b}+1=c$
D. $b+b \cdot \log _{b} a=c$
E. $\frac{\log _{x} b a^{b}}{\log _{x} b}=c$

## Solution

## Answer: D

Justification: Answer A writes the logarithm in exponential form:

$$
\log _{b} b a^{b}=c \Rightarrow b^{c}=b a^{b}
$$

We can also use the property $\log x y=\log x+\log y$ to break up the log:

$$
\log _{b} b a^{b}=\underbrace{\log _{b} b}_{1}+\underbrace{\log _{b} a^{b}}_{b \log _{b} a}
$$

When we let $\log _{b} b=1$, we get answer C.
When we let $\log _{b} a^{b}=b \log _{b} a$, we get answer B.
Answer continues on the next slide

## Solution Continued

Answer $E$ uses the change of base property to rewrite the logarithm with base $x$ :

$$
\log _{b} b a^{b}=\frac{\log _{x} b a^{b}}{\log _{x} b}
$$

Answer D is very similar to answers B and C , although $\log _{b} b \neq b$. Therefore answer D is the only non-equivalent expression:

$$
\log _{b} b a^{b} \neq b+b \log _{b} a
$$

## Properties of Logarithms IV

Which of the following is equivalent to the expression shown:

$$
\frac{\log _{k} 2 b^{a}}{\log _{k} b}
$$

A. $\log _{k} 2 b^{a-1}$
B. $\log _{k} 2 b^{a}-\log _{k} b$
C. $\log _{b} 2 b^{a}$
D. $\log _{2 b^{a}} b$

Change of base property:

$$
\log _{y} x=\frac{\log _{c} x}{\log _{c} y}
$$

E. $2+\mathrm{a}$

## Solution

## Answer: C

Justification: The most common mistake that may arise when answering this question is incorrectly stating:

$$
\frac{\log _{k} 2 b^{a}}{\log _{k} b}=\log _{k} \frac{2 b^{a}}{b}
$$

## This statement is not correct

Making this error will lead to answer A or B.
Instead, we can use the change of base property to write the expression as a single logarithm:

$$
\frac{\log _{k} 2 b^{a}}{\log _{k} b}=\log _{b} 2 b^{a}
$$

Change of base property:

$$
\log _{y} x=\frac{\log _{c} x}{\log _{c} y}
$$

## Properties of Logarithms V

Which of the following are equivalent to the following expression
$(\log (a+b))^{n}$
A. $(\log a+\log b)^{n}$
B. $n \log a+n \log b$
C. $n \log (a+b)$
D. $\log \left(a^{n}+b^{n}\right)$
E. None of the above

## Solution

## Answer: E

Justification: Expressions in the form $\log (a+b)$ and $(\log (a))^{n}$ cannot be simplified. Therefore, $(\log (a+b))^{n}$ cannot be simplified.

The two most common errors are:

$$
\log (a+b)=\log a+\log b \quad(\log a)^{n}=n \log a
$$

Compare the incorrect statements above with the correct logarithm laws:

$$
\log (a b)=\log a+\log b \quad \log \left(a^{n}\right)=n \log (a)
$$

## Properties of Logarithms VI

If $a>b>0$, how do $\log (a)$ and $\log (b)$ compare?
A. $\log a>\log b$
B. $\log a \geq \log b$
C. $\log a<\log b$
D. $\log a \leq \log b$
E. $\log a=\log b$

## Solution

Answer: A
Justification: It may be easier to compare logarithms by writing them as exponents instead. Let:

$$
\log a=A \quad \log b=B
$$

Converting to exponents gives: $10^{A}=a$ and $10^{B}=b$

$$
\begin{array}{cl}
a>b & \text { Given in the question } \\
10^{A}>10^{B} & \text { Substitute } a=10^{A}, b=10^{B}
\end{array}
$$

The larger the exponent on the base 10, the larger our final value. This is true whenever the bases are the same and greater than 1. For example, $10^{4}>10^{3}$.

$$
\begin{array}{cl}
A>B & \\
\text { The exponent } A \text { must be larger than the exponent } B \\
\log a>\log b & \\
\text { Substitute } A=\log a, B=\log b
\end{array}
$$

## Properties of Logarithms VII

What is the value of the following expression?

$$
b^{\log _{b} a}
$$

A. $a$
B. $b$
C. $b^{a}$
D. $a^{b}$
E. Cannot be simplified

## Solution

## Answer: A

Justification: This is one of the logarithm laws:

$$
b^{\log _{b} a}=a
$$

If $\log _{b} a=x, x$ is the exponent to which $b$ must be raised in order to equal $a$ (in order words, $b^{x}=a$ ) by the definition of logarithms.

$$
\log _{\substack{a=x}}^{\substack{\text { equals } a \\ b \text { to the } x}} \longrightarrow \log _{b} \overbrace{b \text { to the }}^{a=\log _{b} a} \log _{b} a
$$

It is important to remember that logarithms return the value of an exponent. When the exponent returned is the original logarithm, we get the property: $b^{\log _{b} a}=a$

## Alternative Solution

## Answer: A

Justification: We can also solve this problem using other properties of logarithms. Let $b^{\log _{b} a}=x$. The value of $x$ will be our answer. If we write this equation using logarithms:

$$
\begin{gathered}
b^{\log _{b} a}=x \\
\log _{b} x=\log _{b} a
\end{gathered}
$$

$$
\begin{gathered}
x=b^{y} \quad \text { In this question, } \\
\log _{b} x=\stackrel{y}{y}
\end{gathered}
$$

We can now use the property $\log _{b} x=\log _{b} a$ if and only if $x=a$.

$$
\begin{aligned}
x & =a \\
b^{\log _{b} a} & =a
\end{aligned}
$$

## Properties of Logarithms VIII

If $a>b>1$ and $c>1$ how do $\log _{a}(c)$ and $\log _{b}(c)$ compare?
A. $\log _{a}(c)>\log _{b}(c)$
B. $\log _{a}(c) \geq \log _{b}(c)$
C. $\log _{a}(c)<\log _{b}(c)$
D. $\log _{a}(c) \leq \log _{b}(c)$
E. $\log _{a}(c)=\log _{b}(c)$

Use the change of base property:

$$
\log _{a}(c)=\frac{\log (c)}{\log (a)} \log _{b} c=\frac{\log (c)}{\log (b)}
$$

## Solution

## Answer: C

Justification: We can use the change of base property to make the two expressions easier to compare:

$$
\log _{a}(c)=\frac{\log (c)}{\log (a)} \log _{b} c=\frac{\log (c)}{\log (b)}
$$

Since $a, b$, and $c$ are greater than $1, \log (a), \log (b)$, and $\log (c)$ are all positive. From the previous question we learned that $\log (a)>\log (b)$ if $a>b$. Since $\log (a)$ is in the denominator, it will produce a smaller fraction than $\log (b)$.

$$
\frac{\log (c)}{\log (a)}<\frac{\log (c)}{\log (b)} \quad \log _{a}(c)<\log _{b}(c)
$$

## Alternative Solution

## Answer: C

Justification: You can also change the question to exponents and compare. Let:

$$
\begin{array}{ll}
\log _{a}(c)=A & \log _{b}(c)=B \\
a^{A}=c & b^{B}=c
\end{array}
$$

If we equate these two equations we get:

$$
a^{A}=b^{B}
$$

Since $a>b$, the exponent A must be smaller than B . If this were not the case, the LHS will have a larger base and exponent than the RHS, so they cannot be equal.

$$
A>B \Rightarrow \log _{a}(c)<\log _{b}(c)
$$

## Properties of Logarithms IX

Given $\log _{3} 7 \approx 1.7712$, what is the approximate value of $\log _{3} 49$ ?
A. $\sqrt{1.7712}=1.3$
B. $(1.7712)^{2}=3.13$
C. $2(1.7712)=3.5424$
D. $7(1.7712)=12.3984$
E. $3^{2}(1.7712)=15.9408$

Write $\log _{3} 49$ in terms of $\log _{3} 7$ using the following property:

$$
\log a^{n}=n \log a
$$

## Solution

## Answer: C

Justification: Without knowing that $\log _{3} 7 \approx 1.7712$, we can still determine an approximate value for $\log _{3} 49$. Since $\log _{3} 27=3$ and $\log _{3} 81=4$, we should expect $\log _{3} 49$ is between 3 and 4 . We can already rule out answers A, D, and E.

To get a better estimate, we can use the properties of logarithms:

$$
\begin{array}{rlr}
\log _{3} 49 & =\log _{3} 7^{2} & \\
& =2 \log _{3} 7 & \text { since } \log a^{n}=n \log a \\
& \approx 2(1.7712) \quad \text { since } \log _{3} 7 \approx 1.7712 \\
& \approx 3.5424 &
\end{array}
$$

## Properties of Logarithms X

If $\frac{\log _{10} m}{\log _{10} n}=1$ what does $\log _{100}\left(\frac{m}{n}\right)$ equal?
A. 0
B. 0.1
C. 1
D. 10
E. 100

Write $\frac{\log _{10} m}{\log _{10} n}=1$ as $\log _{10} m=\log _{10} n$
Press for hint


## Solution

## Answer: A

Justification: If we simplify the equation $\frac{\log _{10} m}{\log _{10} n}=1$, we can find the relationship between $m$ and $n$ :

$$
\begin{aligned}
\frac{\log _{10} m}{\log _{10} n} & =1 \\
\log _{10} m & =\log _{10} n \\
m & =n \\
\frac{m}{n} & =1
\end{aligned}
$$

We can now plug this value into the other expression:

$$
\log _{100}\left(\frac{m}{n}\right)=\log _{100} 1=0 \quad \text { since } \frac{m}{n}=1
$$

## Alternative Solution

## Answer: A

Justification: The solution can also be found by simplifying $\log _{100}\left(\frac{m}{n}\right)$ first, although much more work is required to get a final answer:

$$
\begin{aligned}
\log _{100}\left(\frac{m}{n}\right) & =\log _{100} m-\log _{100} n \\
& =\frac{\log _{10} m}{\log _{10} 100}-\frac{\log _{10} n}{\log _{10} 100} \\
& \text { Change to base } 10 \\
& =\frac{\log _{10} m}{2}-\frac{\log _{10} n}{2} \\
& =\frac{\log _{10} m}{2}-\frac{\log _{10} m}{2} \quad \text { Since } \log _{10} m=\log _{10} n \\
& =0
\end{aligned}
$$

