

a place of mind

FACULTY OF EDUCATION

Department of Curriculum and Pedagogy

Mathematics Number: Logarithms

Science and Mathematics Education Research Group

Supported by UBC Teaching and Learning Enhancement Fund 2012-2014

Logarithms



Introduction to Logarithms

Find the value of x where $2^x = 8$.





- A. x = 2
- B. x = 3
- C. x = 4
- D. x = 8
- E. x = 16

Write 8 as a power with base 2. Exponent law:

 $a^x = a^y$ if and only if x = y

Answer: B

Justification: Try writing the right hand side of the equation as a power of two.

 $8 = 2^3$

Then use the exponent law: $a^x = a^y$ if and only if x = y

 $2^{x} = 8$ $2^{x} = 2^{3}$ x = 3

Introduction to Logarithms II

Find the value of x where $2^x = 18$.

- A. x = 3
- B. x = 4
- C. x = 5
- D. x = 6
- E. None of the above

Answer: E

Justification: We cannot write 18 as a power of 2. This means we cannot solve this question the same way as the previous one.



Although we can't find the exact answer, we can guess the answer is between 4 and 5.

The answer cannot be written as an integer, so the correct answer is E, "None of the above."

In order to answer this question, we will have to learn about logarithms.

Introduction to Logarithms III

If $y = a^x$ then $\log_a y = x$. Notice how the logarithm ($\log_a y$) returns the exponent of a^x .

Using the above definition, what is the value of $\log_2 8$?

- A. $\log_2 8 = 2$
- B. $\log_2 8 = 3$
- C. $\log_2 8 = 4$
- D. $\log_2 8 = 5$
- E. None of the above

Answer: B

Justification: This question is exactly the same as question 1. We must find the value of *x* where $2^x = 8$.

In question 1, we were asked to find the exponent *x* such that $2^x = 8$. This is exactly the same as finding the value of $\log_2 8$.

$$\log_2 8 = 3$$
 since $2^3 = 8$

In general, we can convert between logs and exponents as follows:



A helpful reminder:



Introduction to Logarithms IV

Find the value of x where $2^x = 18$. Express the answer using logarithms.

Hint: Try expressing the question in terms of logarithms.

A. $x = \log_2 18$

$$B. \quad x = \log_{18} 2$$

- C. $x = \log_2 x$
- D. $x = \log_2 y$
- E. None of the above

If $y = a^x$ then $\log_a y = x$.



Press for hint



Answer: A

Justification: Recall how equations in the form $y = a^x$, can be rewritten with logarithms, and vice-versa:

$$y = a^{x}$$

$$2^{x} = 18$$

$$x = \log_{2} 18$$

Notice how applying this to $2^x = 18$ allows us to solve for *x*, which we were not able to do without logarithms:

This is the exact answer to the equation above, so $2^{\log_2 18} = 18$. We will see later that this result can be generalized to a property of logarithms: $b^{\log_b a} = a$

Logarithm Laws

The following is a summary of important logarithm properties:

$$\log ab = \log a + \log b$$
$$\log \frac{a}{b} = \log a - \log b$$
$$\log a^{n} = n \log a$$

$$\log_b a = \frac{\log_c a}{\log_c b}$$

Other useful properties include:

$$\log_b 1 = 0 \qquad \log_b \frac{1}{a} = -\log_b a$$
$$\log_b b = 1 \qquad b^{\log_b a} = a$$

 $\log_b x = \log_b y$ if and only if x = y

Note: When you see log x, it is assumed that the base of the logarithm is 10.

 $\log x = \log_{10} x$

When you see ln x, the base is assumed to be $e \approx 2.72$.

 $\ln x = \log_e x$

Properties of Logarithms I

What is $\log_7 a + \log_7 b - \log_7 c$ written as a single logarithm?

A. $\log_7(a+b-c)$ B. $\log_7(abc)$ C. $\frac{\log_7(ab)}{\log_7 c}$ D. $\log_7\left(\frac{ab}{c}\right)$ E. $\log_7\left(\frac{c}{ab}\right)$

Answer: D

Justification: We will use the two following two properties to simplify the expression:

1.
$$\log ab = \log a + \log b$$
 2. $\log \left(\frac{a}{b}\right) = \log a - \log b$

First combine the terms $\log_7 a + \log_7 b = \log_7 ab$ using the first property 1 shown above:

$$\log_7 a + \log_7 b - \log_7 c = \log_7 ab - \log_7 c$$

Next combine the term that is subtracted using property 2:

$$\log_7 ab - \log_7 c = \log_7 \left(\frac{ab}{c}\right)$$

Does order matter? Would simplifying the subtracted term first change the answer?

Properties of Logarithms II

What is the value of $\log_{10} 100^8$.

- A. 100^8
- B. 10^{16}
- C. 10^8
- D. 16
- E. 8

Answer: D

Justification: Use exponent and logarithm laws to simplify the problem. We can start by pulling the exponent the outside of the logarithm:

$$log_{10} 100^{8} = 8 log_{10} 100 \quad since \quad log_{b} a^{n} = n log_{b} a$$
$$= 8 \cdot 2 \qquad since \quad log_{10} 100 = log_{10} 10^{2} = 2$$
$$= 16$$

The solution tells us that $10^{16} = 100^8$.

Alternative Solution

Answer: D

Justification: Since the logarithm is in base 10, another strategy is to write all terms with a base of 10.

$$log_{10} 100^{8} = log_{10} (10^{2})^{8}$$

= log_{10} 10^{16} since $(a^{x})^{y} = a^{xy}$
= 16 log_{10} 10 since $log_{b} a^{n} = n log_{b} a$
= 16 since $log_{b} b = 1$

Properties of Logarithms III

Which of the following is <u>not</u> equivalent to the following equation, where b > 0, $b \neq 1$?

 $\log_b ba^b = c$

- A. $b^c = ba^b$
- B. $\log_b b + b \cdot \log_b a = c$
- C. $\log_b a^b + 1 = c$
- D. $b + b \cdot \log_b a = c$

E.
$$\frac{\log_x ba^b}{\log_x b} = c$$

Answer: D

Justification: Answer A writes the logarithm in exponential form:

$$\log_b ba^b = c \implies b^c = ba^b$$

We can also use the property $\log xy = \log x + \log y$ to break up the log:

$$\log_{b} ba^{b} = \log_{b} b + \log_{b} a^{b}$$

$$1 \quad b \log_{b} a$$

When we let $\log_{b} b = 1$, we get answer C.

When we let $\log_b a^b = b \log_b a$, we get answer B.

Answer continues on the next slide

Solution Continued

Answer E uses the change of base property to rewrite the logarithm with base *x*:

$$\log_b ba^b = \frac{\log_x ba^b}{\log_x b}$$

Answer D is very similar to answers B and C, although $\log_b b \neq b$. Therefore answer D is the only non-equivalent expression:

 $\log_b ba^b \neq b + b \log_b a$

Properties of Logarithms IV

Which of the following is equivalent to the expression shown:

 $\frac{\log_k 2b^a}{\log_k b}$

- A. $\log_k 2b^{a-1}$
- B. $\log_k 2b^a \log_k b$
- C. $\log_b 2b^a$
- D. $\log_{2b^a} b$
- E. 2 + a

Change of base property:

$$\log_y x = \frac{\log_c x}{\log_c y}$$





Answer: C

Justification: The most common mistake that may arise when answering this question is incorrectly stating:

$$\frac{\log_k 2b^a}{\log_k b} = \log_k \frac{2b^a}{b}$$
 This statement is not correct

Making this error will lead to answer A or B.

Instead, we can use the change of base property to write the expression as a single logarithm:

$$\frac{\log_k 2b^a}{\log_k b} = \log_b 2b^a$$

Change of base property:

$$\log_y x = \frac{\log_c x}{\log_c y}$$

Properties of Logarithms V

Which of the following are equivalent to the following expression

 $(\log(a+b))^n$

- A. $(\log a + \log b)^n$
- B. $n \log a + n \log b$
- C. $n \log(a+b)$
- D. $\log(a^n + b^n)$
- E. None of the above

Answer: E

Justification: Expressions in the form $\log(a+b)$ and $(\log(a))^n$ cannot be simplified. Therefore, $(\log(a+b))^n$ cannot be simplified.

The two most common errors are:

 $\log(a+b) = \log a + \log b \qquad (\log a)^n = n \log a$

Compare the incorrect statements above with the correct logarithm laws:

 $\log(ab) = \log a + \log b \qquad \log(a^n) = n \log(a)$

Properties of Logarithms VI

If a > b > 0, how do log(a) and log(b) compare?

- A. $\log a > \log b$
- B. $\log a \ge \log b$
- C. $\log a < \log b$
- D. $\log a \leq \log b$
- E. $\log a = \log b$

Answer: A

Justification: It may be easier to compare logarithms by writing them as exponents instead. Let:

 $\log a = A$ $\log b = B$

Converting to exponents gives: $10^{A} = a$ and $10^{B} = b$

a > b Given in the question

 $10^{A} > 10^{B}$ Substitute $a = 10^{A}, b = 10^{B}$

The larger the exponent on the base 10, the larger our final value. This is true whenever the bases are the same and greater than 1. For example, $10^4 > 10^3$.

A > BThe exponent A must be larger than the exponent B $\log a > \log b$ Substitute $A = \log a$, $B = \log b$

Properties of Logarithms VII

What is the value of the following expression?

 $b^{\log_b a}$

- A. *a*
- B. *b*
- C. b^a
- D. a^b
- E. Cannot be simplified

Answer: A

Justification: This is one of the logarithm laws:

 $b^{\log_b a} = a$

If $\log_b a = x$, x is the exponent to which b must be raised in order to equal a (in order words, $b^x = a$) by the definition of logarithms.



It is important to remember that logarithms return the value of an *exponent*. When the exponent returned is the original logarithm, we get the property: $b^{\log_b a} = a$

Alternative Solution

Answer: A

Justification: We can also solve this problem using other properties of logarithms. Let $b^{\log_b a} = x$. The value of *x* will be our answer. If we write this equation using logarithms:

$$b^{\log_b a} = x$$

 $\log_b x = \log_b a$
 $x = b^y$
 $\log_b x = \log_b a$
 $\log_b x = y$
In this question,
 $y = \log_b a$

We can now use the property $\log_b x = \log_b a$ if and only if x = a.

x = a

 $b^{\log_b a} = a$

Properties of Logarithms VIII

If a > b > 1 and c > 1 how do $\log_a(c)$ and $\log_b(c)$ compare?

- A. $\log_a(c) > \log_b(c)$
- B. $\log_a(c) \ge \log_b(c)$
- C. $\log_a(c) < \log_b(c)$
- D. $\log_a(c) \le \log_b(c)$
- E. $\log_a(c) = \log_b(c)$

Use the change of base property:

$$\log_a(c) = \frac{\log(c)}{\log(a)} \quad \log_b c = \frac{\log(c)}{\log(b)}$$



Answer: C

Justification: We can use the change of base property to make the two expressions easier to compare:

$$\log_{a}(c) = \frac{\log(c)}{\log(a)} \quad \log_{b} c = \frac{\log(c)}{\log(b)}$$

Since *a*, *b*, and *c* are greater than 1, log(a), log(b), and log(c) are all positive. From the previous question we learned that log(a) > log(b) if a > b. Since log(a) is in the denominator, it will produce a smaller fraction than log(b).

$$\frac{\log(c)}{\log(a)} < \frac{\log(c)}{\log(b)} \quad \Longrightarrow \quad \log_a(c) < \log_b(c)$$

Alternative Solution

Answer: C

Justification: You can also change the question to exponents and compare. Let:

$$\log_{a}(c) = A \quad \log_{b}(c) = B$$
$$a^{A} = c \qquad b^{B} = c$$

If we equate these two equations we get:

$$a^A = b^B$$

Since a > b, the exponent A must be smaller than B. If this were not the case, the LHS will have a larger base and exponent than the RHS, so they cannot be equal.

$$A > B \implies \log_a(c) < \log_b(c)$$

Properties of Logarithms IX

Given $\log_3 7 \approx 1.7712$, what is the approximate value of $\log_3 49$?

- A. $\sqrt{1.7712} = 1.3$
- B. $(1.7712)^2 = 3.13$
- C. 2(1.7712) = 3.5424
- D. 7(1.7712) = 12.3984
- E. $3^2(1.7712) = 15.9408$

Write $\log_3 49$ in terms of $\log_3 7$ using the following property:

 $\log a^n = n \log a$



Answer: C

Justification: Without knowing that $\log_3 7 \approx 1.7712$, we can still determine an approximate value for $\log_3 49$. Since $\log_3 27 = 3$ and $\log_3 81 = 4$, we should expect $\log_3 49$ is between 3 and 4. We can already rule out answers A, D, and E.

To get a better estimate, we can use the properties of logarithms:

$$log_{3} 49 = log_{3} 7^{2}$$

= 2 log_{3} 7 since log $a^{n} = n \log a$
 $\approx 2(1.7712)$ since log_{3} 7 ≈ 1.7712
 ≈ 3.5424

Properties of Logarithms X

If
$$\frac{\log_{10} m}{\log_{10} n} = 1$$
 what does $\log_{100} \left(\frac{m}{n}\right)$ equal?

- **B.** 0.1
- **C.** 1
- D. 10
- E. 100

Write
$$\frac{\log_{10} m}{\log_{10} n} = 1$$
 as $\log_{10} m = \log_{10} n$





Answer: A

Justification: If we simplify the equation $\frac{\log_{10} m}{\log_{10} n} = 1$, we can find the relationship between *m* and *n*:

$$\frac{\log_{10} m}{\log_{10} n} = 1$$
$$\log_{10} m = \log_{10} n$$
$$m = n$$
$$\frac{m}{n} = 1$$

We can now plug this value into the other expression:

$$\log_{100}\left(\frac{m}{n}\right) = \log_{100} 1 = 0 \qquad \text{since } \frac{m}{n} = 1$$

Alternative Solution

Answer: A

Justification: The solution can also be found by simplifying $\log_{100}\left(\frac{m}{n}\right)$ first, although much more work is required to get a final answer:

$$\log_{100}\left(\frac{m}{n}\right) = \log_{100} m - \log_{100} n$$

= $\frac{\log_{10} m}{\log_{10} 100} - \frac{\log_{10} n}{\log_{10} 100}$ Change to base 10
= $\frac{\log_{10} m}{2} - \frac{\log_{10} n}{2}$ $\log_{10} 100 = 2$
= $\frac{\log_{10} m}{2} - \frac{\log_{10} m}{2}$ Since $\log_{10} m = \log_{10} n$
= 0