a place of mind

FACULTY OF EDUCATION

Department of
Curriculum and Pedagogy

Mathematics

Probability: Permutations

Science and Mathematics
Education Research Group

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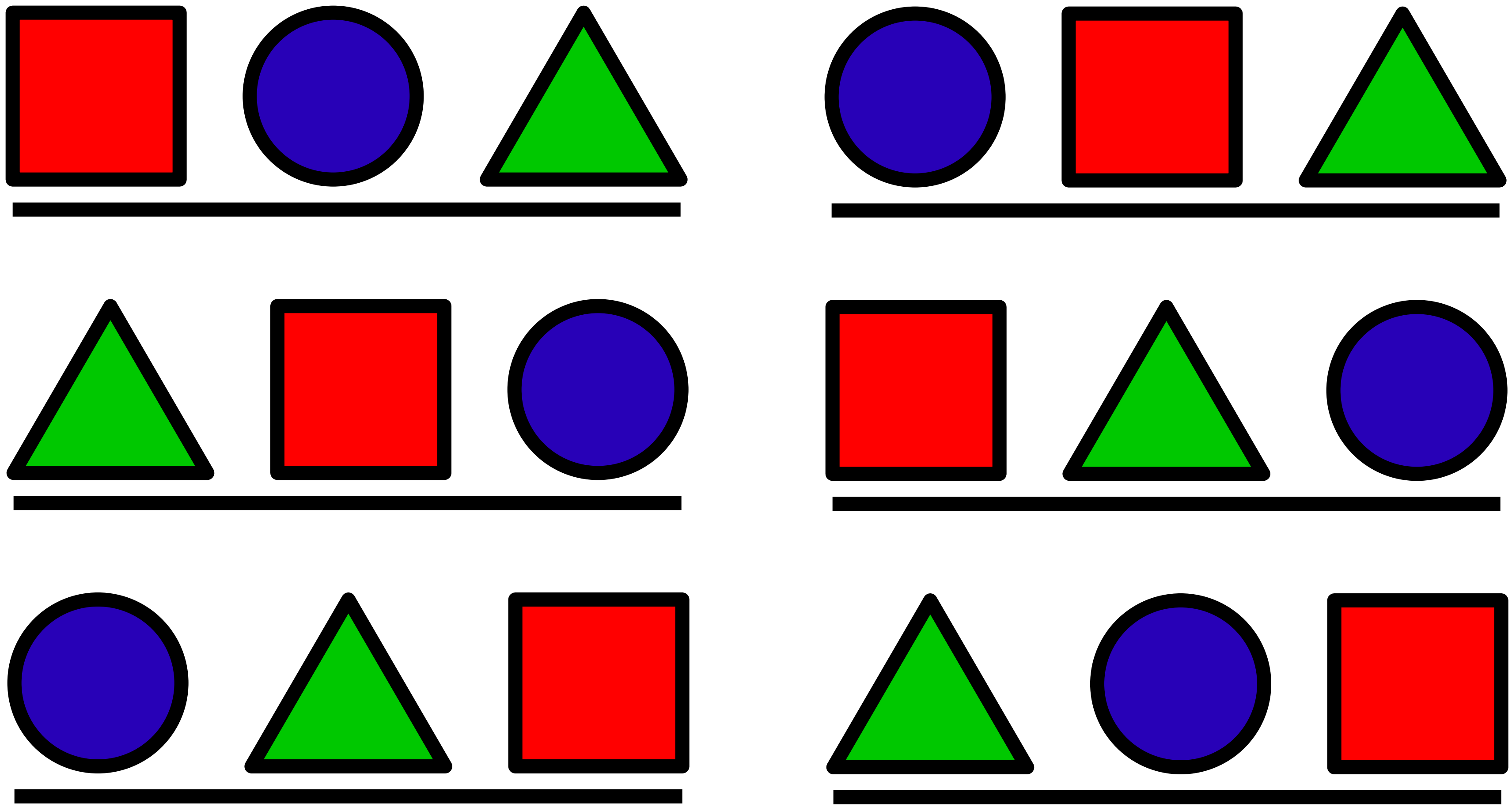
Permutations



Permutations I

A group of 10 children race around a track. How many different ways can these 10 children place, assuming that there were no ties.

One possible placement after the race:

- *1st: Alex*
- *2nd: Bob*
- *3rd: Christy*
- *4th: David*
- *5th: Elizabeth*
- *6th: Fred*
- *7th: George*
- *8th: Heather*
- *9th: Ian*
- *10th: Jeremy*

A. 10^{10}

B. 10^{9}

C. 10^{2}

D. 10

E. $10(9)(8)\ldots(3)(2)(1) = 10!$

Solution

Answer: E

Justification: Any of the 10 children can win, so we have 10 different choices for first place. This will leave us with only 9 choices for the 2^{nd} place finish, 8 choices for 3^{rd} place, and so on.

Choices:

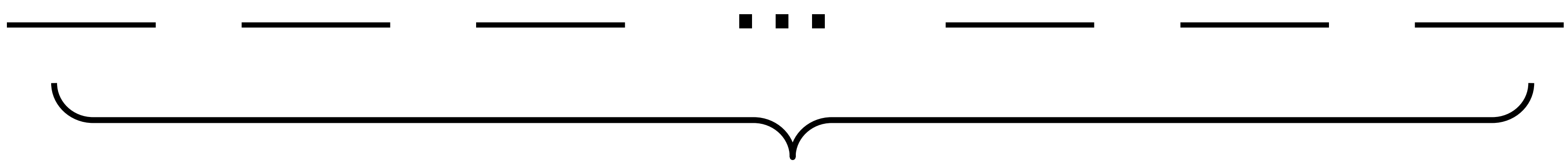
10	9	8	...	3	2	1
1^{st}	2^{nd}	3^{rd}		8^{th}	9^{th}	10^{th}

The number of ways the children can place is therefore:

$$10(9)(8)...(3)(2)(1) = 10! = 3628800$$

Permutations II

Consider a group of n different people. Which of the expressions equals the number of different ways the people can line up?



n positions

A. n^n

B. n^{n-1}

C. n^2

D. n

E. $n(n-1)(n-2)...(2)(1) = n!$

Solution

Answer: E

Justification: There are n choices to choose the person at the front of the line. This leaves n-1 people to choose for the second person in the line. Continuing to place the remaining people down the line gives:

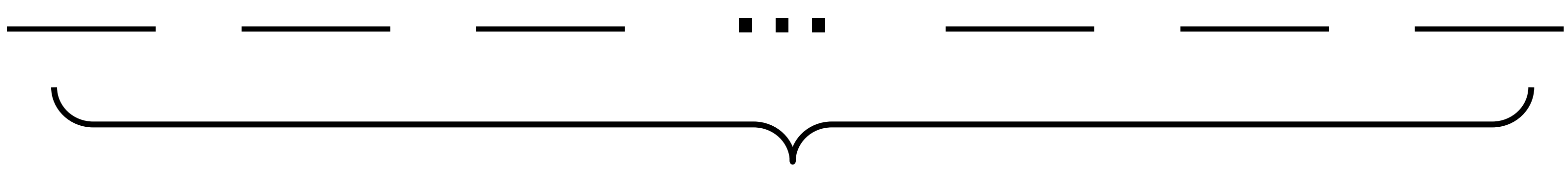
n	n-1	n-2	...	3	2	1

Since these choices are done sequentially, the number of choices are multiplied together:

Total # of arrangements = $n(n-1)(n-2)...(2)(1) = n!$

Permutations III

Now consider when there are only r slots in a lineup, where r ≤ n. How many ways can n people be lined up in this shorter line?



r positions for n people

A. $n(n-1)(n-2)...(2)(1)$

B. $n(n-1)(n-2)...(n-r)$

C. $n(n-1)(n-2)...(n-r+1)$

D. $n(n-1)(n-2)...(n-r-1)$

E. $(n-r)!$

Solution

Answer: C

Justification: The number of choices for each position is shown below:

1st position: n choices
2nd position: (n-1) choices
3rd position: (n-2) choices
rth position: (n-(r-1)) choices

Since these choices are done sequentially, the number of choices are multiplied together:

Total # of arrangements = n(n-1)(n-2)...(n-r+1)

n	n-1	n-2	...	n-r+3	n-r+2	n-r+1

Permutations IV

Let the number of ways we can arrange a set of n objects into r slots be denoted by ${}_nP_r$. From the last question, we learned that:

$${}_nP_r = n(n-1)(n-2)...(n-r+1)$$

Using factorial notation, ${}_nP_r$ can be written as:

A. $(n-r)!$

B. $(n-r+1)!$

C. $\dfrac{n!}{(n-r)!}$

D. $\dfrac{n!}{(n-r-1)!}$

E. $\dfrac{n!}{(n-r+1)!}$

Solution

Answer: C

Justification: ${}_nP_r = n(n\text{-}1)(n\text{-}2)...(n\text{-}r\text{+}1)$

The formula begins multiplying at n, then starts decreasing by 1. We must therefore start with n!:

$$n! = n(n-1)!$$

Expanding until we reach (n-r+1) gives:

$$n! = n(n-1)(n-2)...(n-r+1)(n-r)!$$

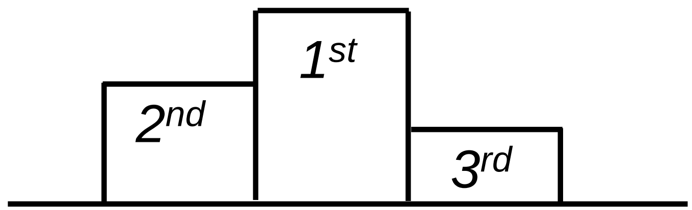
Dividing both sides by (n-r)!: ${}_nP_r = \dfrac{n!}{(n-r)!} = n(n-1)(n-2)...(n-r+1)$

For example: ${}_3P_3 = \dfrac{3!}{(3-3)!} = 3(2)(1) = 6$

(ABC, ACB, BAC, BCA, CAB, CBA)

Permutations V

A group of 10 children race around a track. This time, the tournament is only concerned about who comes in first, second and third. How many ways can the 1st, 2nd, and 3rd place be handed out amongst the children?



Alex
Bob
Christy
David
Elizabeth
Fred
George
Heather
Ian
Jeremy

A. $_{10}P_{10}$

B. $_{10}P_{7}$

C. $_{10}P_{3}$

D. 10!

E. 3!

Solution

Answer: B

Justification: There are 10 children to arrange into 3 positions. Using the formula with n = 10 and r = 3:

$$ {}_{10}P_3 = \frac{10!}{(10-3)!} = \frac{10!}{7!} = 10(9)(8) = 720 $$

Alternative solutions:

There are 10 ways to choose the first place winners. After this child is chosen, there are only 9 choices left for 2^{nd} place, and 8 choices for 3^{rd} place. Therefore the total number of ways is:

10(9)(8) = 720

Permutations VI

What is the value of ${}_{99999}P_1$?

A. 0
B. 1
C. 99999
D. 99999(99998)(99997)….(3)(2)(1)
E. A number too large to simplify

Solution

Answer: C

Justification: ${}_{99999}P_1$ represents the number of ways we can arrange 99999 objects into 1 slot. This is the same as the number of ways we can pick 1 object out of 99999 objects. Therefore, ${}_{99999}P_1$ = 99999.

Alternative solution:

$${}_{99999}P_3 = \frac{99999!}{(99999-1)!} = \frac{99999(99998!)}{99998!} = 99999$$

Permutations VII

Consider the permutations of the letters AAAABC.

If the 6 letters were unique, there would be 6! permutations. However, the 4 A's can be organized in 4! different ways, giving the same permutation. How many ways can AAAABC be organized?

A. $6!-4!$

B. $(6-4)!$

C. $\frac{6!}{4!}$

D. $\frac{6!}{(6-4)!}$

E. $\frac{6!}{6!-4!}$

Solution

Answer: C

Justification: Out of the 6! different orders of 6 unique letters, all the permutations with different orderings of the 4 A's must be removed. This is done by dividing the total number of permutations by the number of permutations of the repeated letters (here, the As).

For example, consider the permutations of ABC and AAC:

Permutations of ABC = (ABC, ACB, BAC, BCA, CAB, CBA) = 3!

Permutations of AAC = (AAC, ACA, AAC, ACA, CAA, CAA)

$$= (AAC, ACA, CAA) = \frac{3!}{2}$$

Permutations VIII

Which of the following set of letters has the most permutations? Remember:

$$ {}_nP_r = \frac{n!}{(n-r)!} = n(n-1)(n-2)...(n-r+1) $$

A. ABCD

B. AABBC

C. AAABB

D. AAAAAAB

E. AAABBB

Solution

Answer: B

Justification:

$$ABCD: \quad 4! = 4(3)(2) = 24$$

$$AABBC: \quad \frac{5!}{2!2!} = \frac{5(4)(3)}{2} = 30$$

$$AAABB: \quad \frac{5!}{3!2!} = \frac{5(4)}{2} = 10$$

$$AAAAAAB: \quad \frac{7!}{6!} = 7$$

$$AAABBB: \quad \frac{6!}{3!3!} = \frac{6(5)(4)}{6} = 20$$

Permutations IX

Which of the following can be done in the most number of ways? Remember:

$$ {}_nP_r = \frac{n!}{(n-r)!} = n(n-1)(n-2)...(n-r+1) $$

A. Arranging 10 unique objects into 10 ordered slots

B. Arranging 10 unique objects into 9 ordered slots

C. Arranging 10 unique objects into 10 unordered slots

D. A = B > C

E. A = B = C

Solution

Answer: D

Justification:

When arranging 10 objects into 10 ordered slots, only the first 9 slots add to the number of permutations. After 9 objects have been placed, the last object must go into the last slot. Notice that the formulas give the same result:

$$_{10}P_{10} = \frac{10!}{(10-10)!} = 10! \quad \text{since } 0! = 1$$

$$_{10}P_{9} = \frac{10!}{(10-9)!} = 10! \quad \text{since } 1! = 1$$

If the slots were unordered, then we are asked to find the number of ways we can choose 10 objects out of 10 objects. There is only 1 way we can do this. So statement C = 1, which is less than both A and B.