a place of mind

FACULTY OF EDUCATION

Department of
Curriculum and Pedagogy

Mathematics
Quadratic Formula

Science and Mathematics
Education Research Group

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The Quadratic Formula

$$x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$$

Quadratic Formula Part I

Write the equation $ax^2+bx+c=0$ in the form $a(x-p)^2+q=0$

$$ax^2+bx+c=a\left(x^2+\frac{b}{a}x\right)+c$$ Factor out *a*

$$=a\left(x^2+\frac{b}{a}x+\frac{b^2}{4a^2}-\frac{b^2}{4a^2}\right)+c$$ Complete the square

$$=a\left(x+\frac{b}{2a}\right)^2-\frac{b^2}{4a}+c$$

$$=a\left(x+\frac{b}{2a}\right)^2-\frac{b^2-4ac}{4a}$$

$$p=\frac{-b}{2a},\quad q=-\frac{b^2-4ac}{4a}$$

Quadratic Formula Part II

Solve $a\left(x+\frac{b}{2a}\right)^2-\frac{b^2-4ac}{4a}=0$ for *x:*

$$a\left(x+\frac{b}{2a}\right)^2-\frac{b^2-4ac}{4a}=0$$

$$\left(x+\frac{b}{2a}\right)^2=\frac{b^2-4ac}{4a^2},\quad a\neq 0$$ Divide by *a*

$$x+\frac{b}{2a}=\frac{\pm\sqrt{b^2-4ac}}{2a}$$ Square root both sides

$$x=\frac{-b\pm\sqrt{b^2-4ac}}{2a}$$ *The Quadratic Formula*

Quadratic Formula I

What are the roots of the quadratic equation:

$$2x^2 - x - 6 = 0\ ?$$

A. $x = 2, \quad -\frac{2}{3}$

B. $x = -2, \quad \frac{2}{3}$

C. $x = 2, \quad -\frac{3}{2}$

D. $x = \frac{-1+\sqrt{47}}{4}, \quad \frac{-1-\sqrt{47}}{4}$

E. $x = \frac{-1+\sqrt{48}}{4}, \quad \frac{-1-\sqrt{48}}{4}$

Remember:

$$x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$$

Solution

Answer: C

Justification: Before applying the quadratic formula, always check if the equation can be factored.

Factoring:

$$2x^2 - x - 6 = 0$$

$$(2x+3)(x-2) = 0$$

$$x = -\frac{3}{2}, \quad 2$$

It is much faster to solve this question by factoring rather than using the quadratic formula.

Quadratic formula:

$$x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}, \quad a = 2, b = -1, c = -6$$

$$x = \frac{-(-1) \pm \sqrt{(-1)^2 - 4(2)(-6)}}{2(2)}$$

$$x = \frac{1 \pm 7}{4}$$

$$x = 2, \quad -\frac{3}{2}$$

Quadratic Formula II

What are the roots of the equation:

$x^2 + 3x + 1 = 0$?

$$x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$$

A. $x = \frac{-3}{2} + \frac{\sqrt{5}}{2}$

B. $x = \frac{-3}{2} + \frac{\sqrt{5}}{2}, \quad \frac{-3}{2} - \frac{\sqrt{5}}{2}$

C. $x = -3 + \frac{\sqrt{5}}{2}$

D. $x = -3 + \frac{\sqrt{5}}{2}, \quad -3 - \frac{\sqrt{5}}{2}$

E. Cannot be determined

Solution

Answer: B

Justification: First check if the quadratic can be factored. In this case it cannot, so apply the quadratic formula:

$$x^2 + 3x + 1 = 0$$

$$x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}, \quad a = 1, b = 3, c = 1$$

$$x = \frac{-3 \pm \sqrt{3^2 - 4(1)(1)}}{2(1)}$$

$$x = \frac{-3 \pm \sqrt{5}}{2}$$

$$x = -\frac{3}{2} + \frac{\sqrt{5}}{2}, \quad -\frac{3}{2} - \frac{\sqrt{5}}{2}$$

Quadratic Formula III

Consider the part of the quadratic formula inside the square root. This is known as the discriminant, $\Delta = b^2 - 4ac$

If a quadratic equation has 1 solution, what can be concluded about the discriminant?

A. $\Delta < 0$

B. $\Delta \leq 0$

C. $\Delta = 0$

D. $\Delta = 1 \quad or \quad \Delta = -1$

E. $\Delta > 0$

$$x = \frac{-b \pm \sqrt{\Delta}}{2a}, \quad \Delta = b^2 - 4ac$$

Solution

Answer: C

Justification: If a quadratic equation has 1 solution, then the quadratic formula must also give 1 value for x.

$$x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$$

This can only happen in the case where:

$$x = \frac{-b \pm 0}{2a} = \frac{-b}{2a}$$

If $\Delta = b^2 - 4ac = 0,$ then the corresponding quadratic equation will have 1 solution.

Quadratic Formula IV

The discriminant of a quadratic is negative $(\Delta < 0)$.

How many real solutions does the corresponding quadratic equation have?

A. 0, 1, or 2 real solutions

B. 0 or 1 real solution(s)

C. 2 real solutions

D. 1 real solution

E. 0 real solutions

Solution

Answer: E

Justification: If the quadratic formula were used to determine the solutions to a quadratic equation, we would get

$$x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$$

$$x = \frac{-b \pm \sqrt{\Delta}}{2a}, \quad \Delta = b^2 - 4ac < 0$$

If the discriminant is negative, we would get a negative value inside the square root. The quadratic equation therefore has no real solutions.

Quadratic Formula V

How many real solutions does the following quadratic equation have?

$$27x^2 - 14x - 13 = 0$$

A. 2 real solutions
B. 1 real solution
C. 0 real solutions
D. Infinite solutions
E. Cannot be determined

Solution

Answer: A

Justification: Determine if the discriminant is greater than zero, equal to zero, or less than zero:

$$\Delta = b^2 - 4ac$$

$$\Delta = (-14)^2 - 4(27)(-13)$$

$$\Delta = 14^2 + 4(27)(13) > 0$$

$$\Delta > 0$$

It is not necessary to calculate the exact value of the discriminant. It is clear from the above calculation that the discriminant will be the sum of 2 positive values. Since the discriminant is positive, the quadratic equation will have 2 solutions.

Quadratic Formula VI

After using the quadratic formula to solve a quadratic equation, the following 2 solutions are found:

$$x = \frac{-5 \pm \sqrt{13}}{6}$$

What is the quadratic equation with this solution?

A. $3x^2 + 5x + 13 = 0$

B. $3x^2 - 5x + 13 = 0$

C. $3x^2 + 5x + 1 = 0$

D. $6x^2 - 5x + 1 = 0$

E. $6x^2 + 5x + 1 = 0$

Solution

Answer: C

Justification: Compare the solution with the quadratic formula:

$$x = \frac{-5 \pm \sqrt{13}}{6} \qquad x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$$

From the above, we can see that -*b* = -5 and 2*a* = 6. After we find the values of *a* and *b,* we can use the discriminant to find *c*:

$$2a = 6 \quad -b = -5 \qquad \Delta = b^2 - 4ac$$

$$a = 3 \quad b = 5 \qquad 13 = 5^2 - 4(3)c \quad \Delta = 13, a = 3, b = 5$$

$$c = 1$$

Since we now know *a*, *b* and *c*, the quadratic equation is therefore:

$$3x^2 + 5x + 1 = 0$$

Quadratic Formula VII

Consider a quadric function $f(x) = ax^2 + bx + c$, where $a > 0$ and $b^2 - 4ac < 0$.

Which of one of the following statements is true about function?

A. $f(x) > 0$ for all x

B. $f(x) \geq 0$ for all x

C. $f(x) < 0$ for all x

D. $f(x) \leq 0$ for all x

E. None of the above

Solution

Answer: A

Justification: Since the discriminant is negative, the quadratic equation $0 = ax^2 + bx + c$

has no solution. This means that the quadratic function never crosses the x-axis.

$$f(x) = ax^2 + bx + c \neq 0$$

The quadratic function must either lie completely above or completely below the x-axis. Since we also know that $a > 0$, the quadratic opens upwards. Therefore, in order to never cross the x-axis, the graph must lie above the x-axis for all values of x.

$$f(x) > 0$$

Quadratic Formula VIII

Recall that a quadratic function $f(x) = a\left(x + \frac{b}{2a}\right)^2 - \frac{b^2 - 4ac}{4a}$ has a vertex at $\left(\frac{-b}{2a}, \frac{-\Delta}{4a}\right)$.

If $a < 0$ and $\Delta < 0$, which one of the following statements is true?

A. $f(x)$ opens up, vertex above x-axis

B. $f(x)$ opens up, vertex below x-axis

C. $f(x)$ opens down, vertex above x-axis

D. $f(x)$ opens down, vertex below x-axis

E. $f(x)$ opens down, vertex on the x-axis

Solution

Answer: D

Justification: If $a < 0$, then $a\left(x + \frac{b}{2a}\right)^2 \leq 0$, so $f(x) \leq \frac{-\Delta}{4a}$.

The parabola opens downwards.

When $a < 0$ and $\Delta < 0$ the y-value of the vertex $-\Delta/4a$ is negative. The vertex is therefore below the x-axis.

If the quadratic function opens downwards and the vertex is below the x-axis, it will never cross the x-axis and therefore has no solutions.

This agrees with the conclusion that when $\Delta < 0$, the quadratic equation has no solutions.

Summary

The table below shows the connection between the discriminant and graphs of quadratic functions.

$\Delta > 0,\quad 2$ solutions

$\Delta = 0,\quad 1$ solution

$\Delta < 0,\quad 0$ solutions

a	Δ	$\frac{-\Delta}{4a}$	**Number of solutions**
+	+	-	2 solutions - Opens up and vertex below x-axis
+	-	+	0 solutions - Opens up and vertex above x-axis
-	+	+	2 solutions - Opens down and vertex above x-axis
-	-	-	0 solutions - Opens down and vertex below x-axis
+	0	0	1 solution – Opens up and vertex on x-axis
-	0	0	1 solution – Opens down and vertex on x-axis