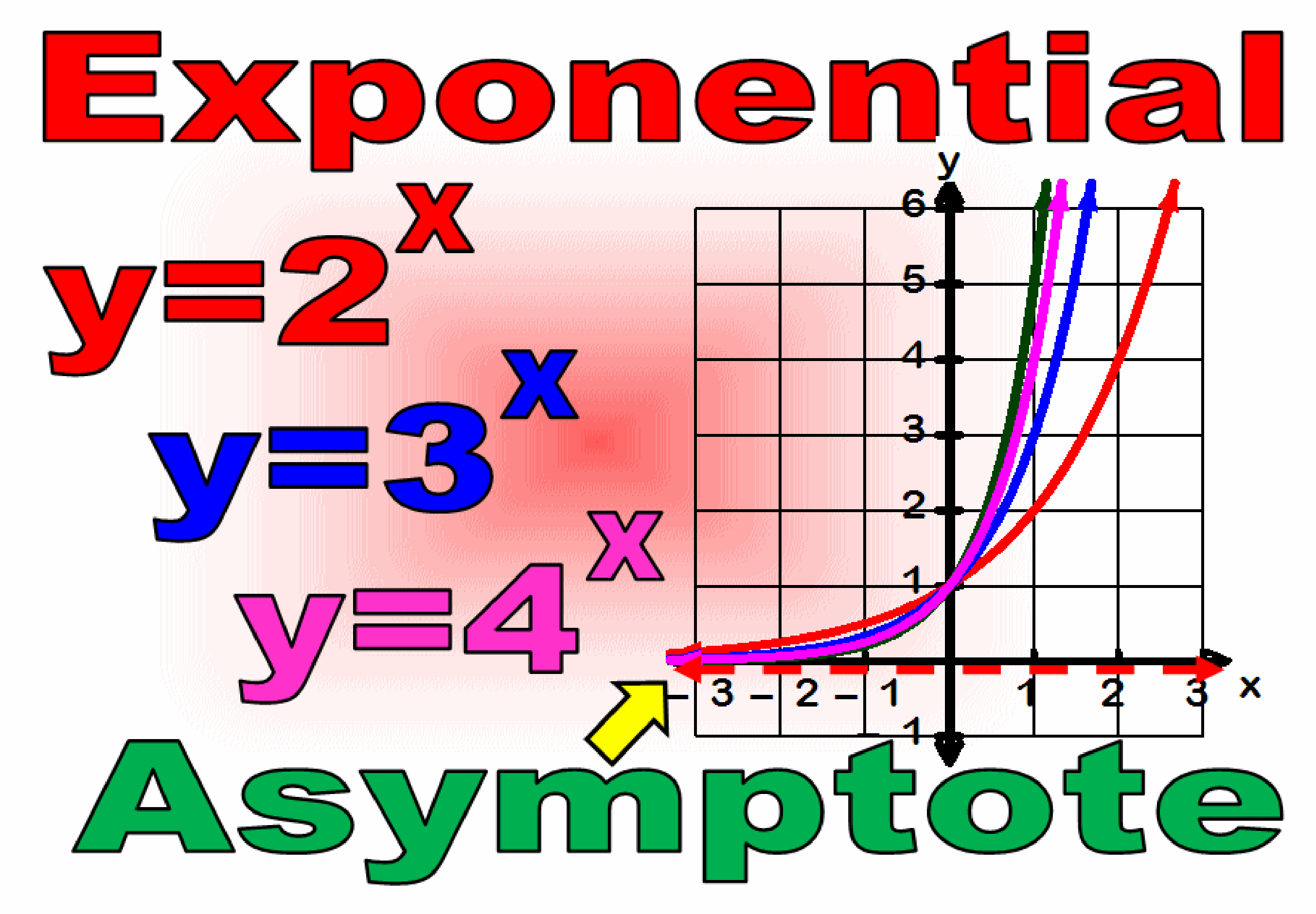
a place of mind
FACULTY OF EDUCATION
Department of
Curriculum and Pedagogy

Mathematics
Functions and Relations: Exponential Functions

Science and Mathematics
Education Research Group

Supported by UBC Teaching and Learning Enhancement Fund 2012-2015

Exponential Functions



Retrieved from http://maths.nayland.school.nz/Year_12/AS_2.2_Graphs/8_exponential.htm

Exponential Functions I

What is the domain and range for this exponential function?

$$y = 2^x$$

A. $\{x \mid x \in R\}, \{y \mid y \geq 2, y \in Z\}$

B. $\{x \mid x \in R\}, \{y \mid y \geq 0, y \in Z\}$

C. $\{x \mid x \in R\}, \{y \mid y \geq 0, y \in R\}$

D. $\{x \mid x \in R\}, \{y \mid y > 0, y \in R\}$

E. $\{x \mid x \in R\}, \{y \mid y \in R\}$

Solution

Answer: D

Justification: For $y = 2^x$, there is no restriction that prohibits what x could be. Therefore, the domain of x is: $x \in R$.

For our range, when x is a positive number, y should also be positive.

When x = 0, y is 1 as the exponents rule: $y^0 = 1, y \neq 0$

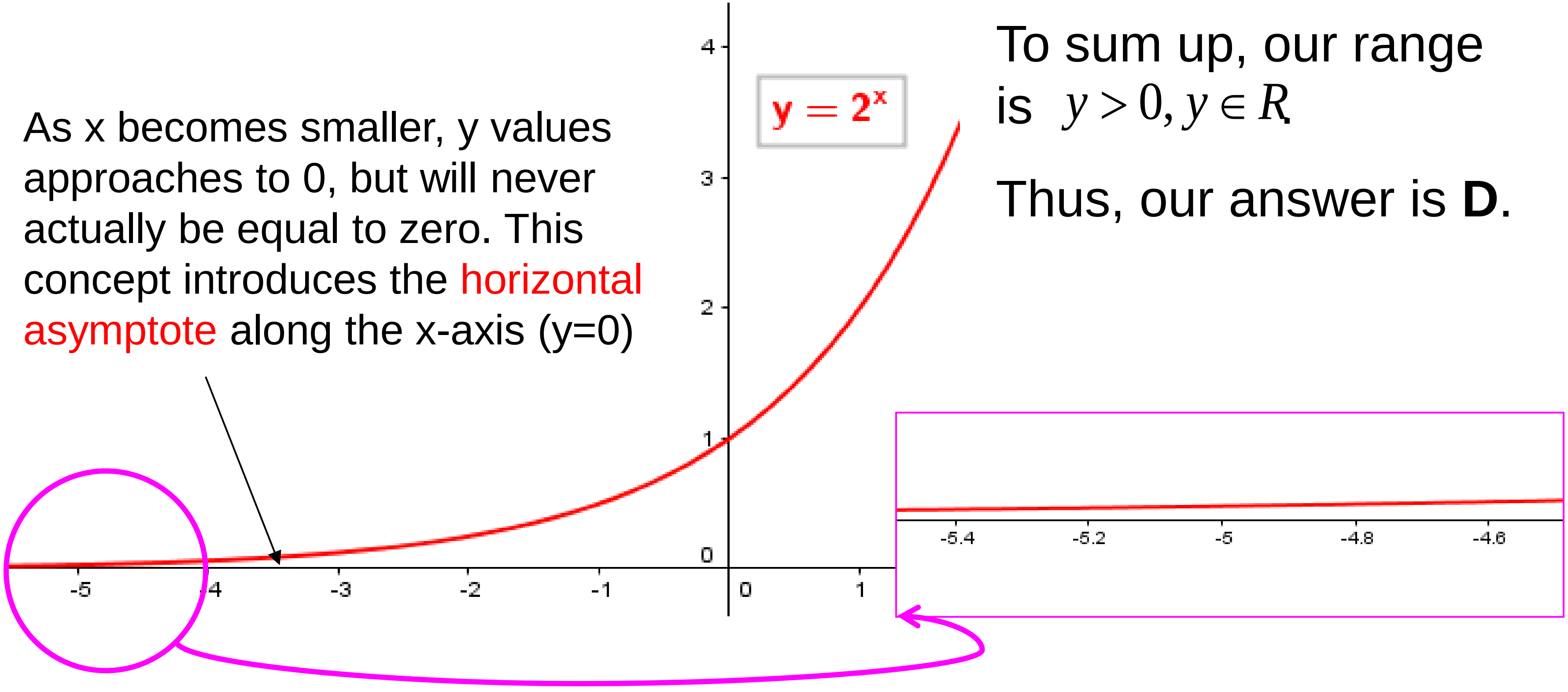
We can also see that there is no value of x that will give us y = 0.

When x is negative y value will never be negative as: $x^{-n} = \frac{1}{x^n}$

Solution Continued

Thus, since our y values will always be greater than 0 for all x values, we know that all the y values for $y = 2^x$ will always be above the x axis, creating the horizontal asymptote of y = 0.

As x becomes smaller, y values approaches to 0, but will never actually be equal to zero. This concept introduces the horizontal asymptote along the x-axis (y=0)



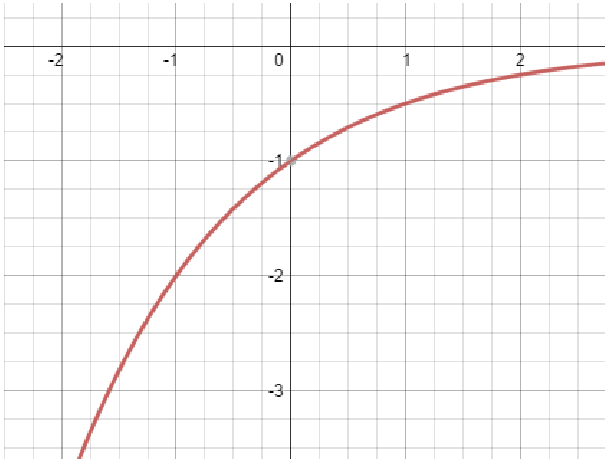
To sum up, our range is $y > 0, y \in R$

Thus, our answer is **D**.

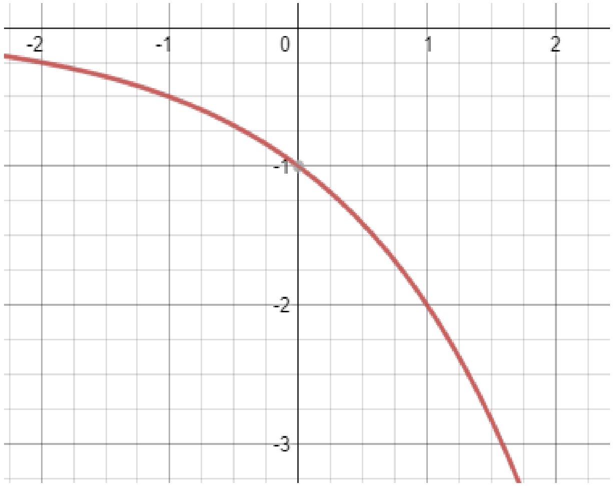
Exponential Functions II

Which of the following graphs corresponds to $f(x) = 0.5^x$?

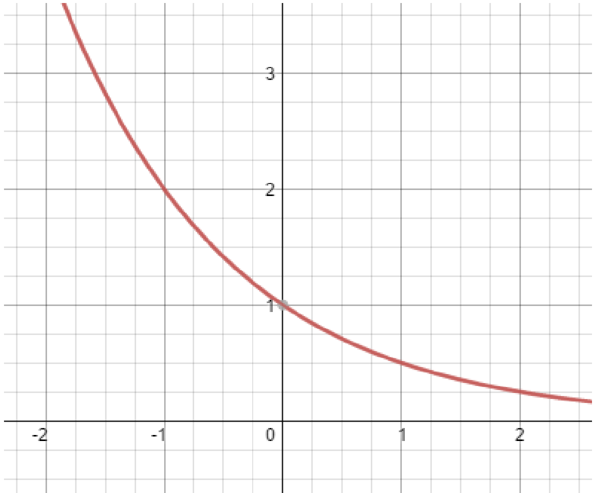
A.



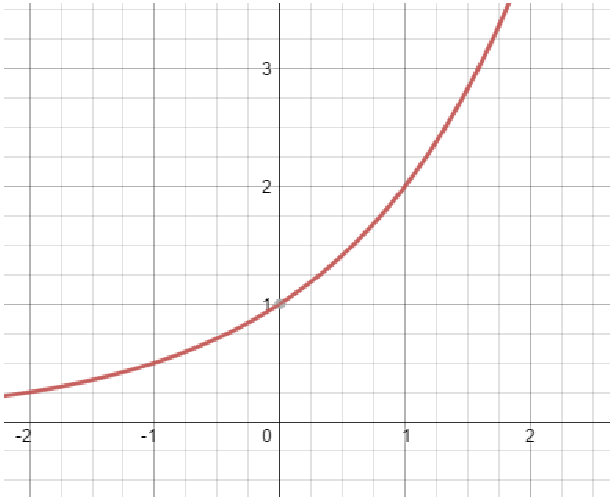
B.



C.



D.



Solution

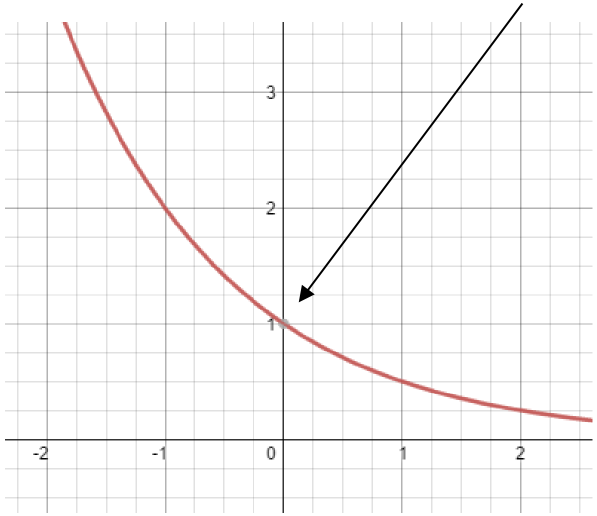
Answer: C

Justification: From the previous problem, we know that there is no x-intercept. Our y-intercept is (0,1).

y-intercept

$f(0) = 0.5^0$

$= 1$



Note: $0.5 = \frac{1}{2}$ and so

$f(x) = 0.5^x = \frac{1^x}{2^x} = \frac{1}{2^x} = 2^{-x}$

When $x \to -\infty, y \to \infty$ and also when $x \to \infty, y \to 0$.

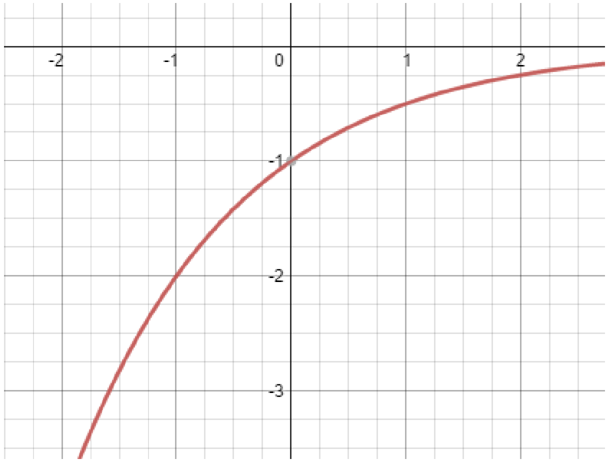
approaches

Now we know that as x increases, then y decreases. The only trend that displays these two facts is C. Thus, our answer is C.

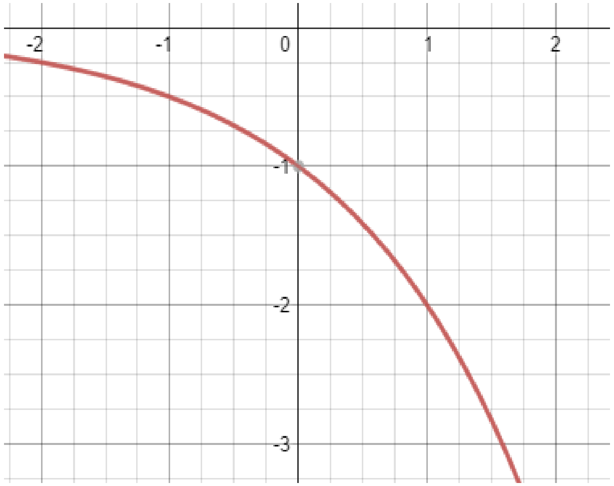
Exponential Functions III

Which of the following graphs corresponds to $f(x) = 0.5^{-x}$?

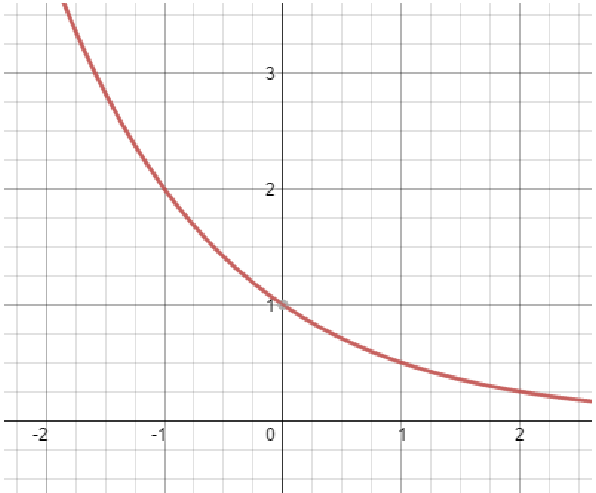
A.



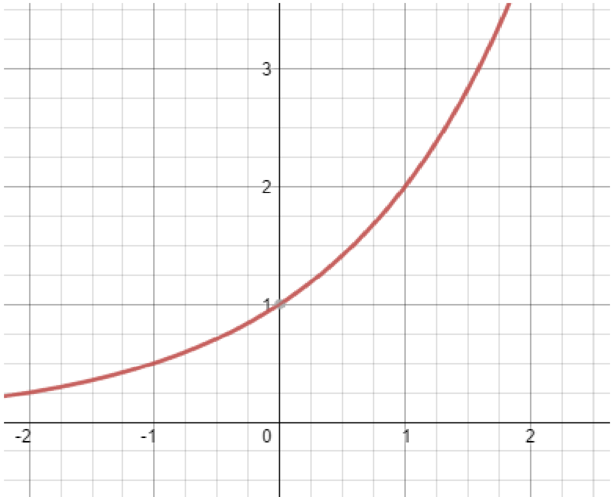
B.



C.



D.



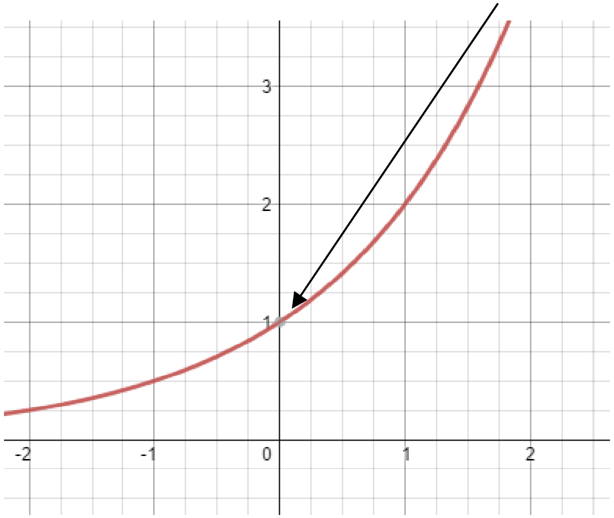
Solution

Answer: D

Justification: From the previous problem, we know that there is no x-intercept. Our y-intercept is (0,1).

y-intercept

$$f(0) = 0.5^0$$
$$= 1$$



Note: $0.5 = \frac{1}{2}$ and so

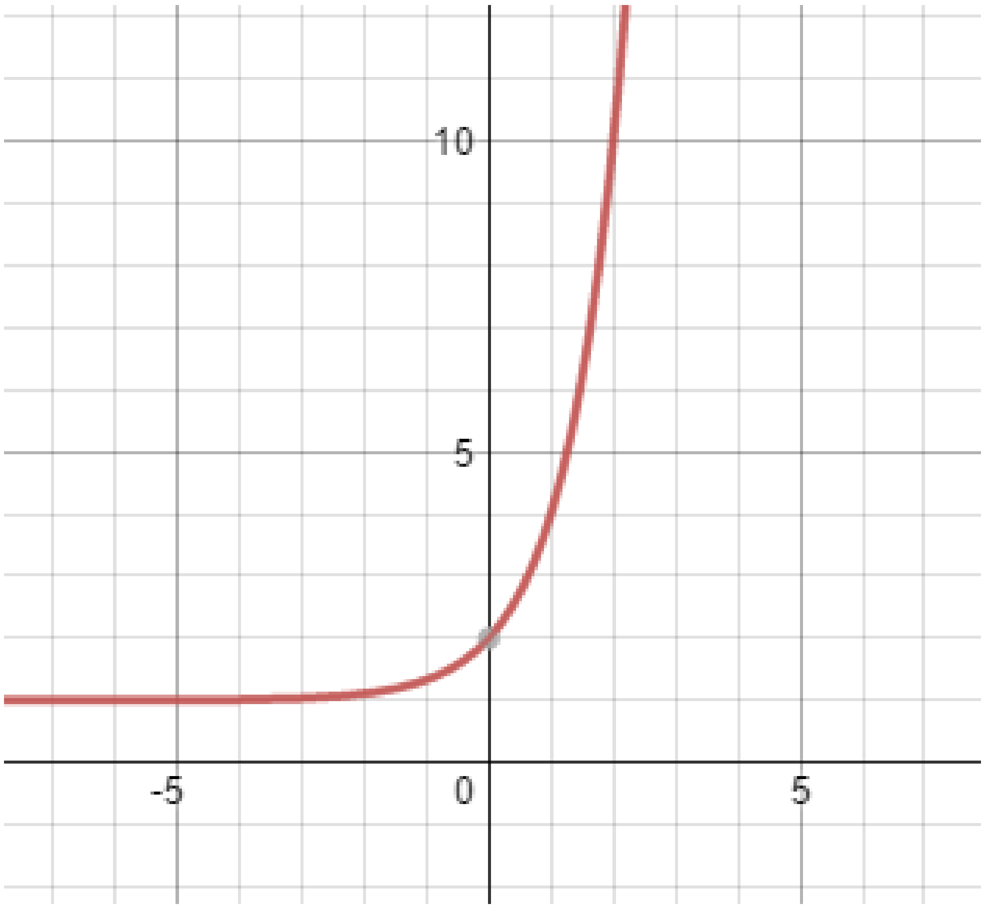
$$f(x) = 0.5^{-x} = \frac{1^{-x}}{2^{-x}} = \frac{1}{2^{-x}}$$
$$f(x) = 2^x$$

When $x \to -\infty, y \to 0$ and also when $x \to \infty, y \to \infty$.

Now we know that as x increases, y increases as well. The only trend that displays these two facts is D. Thus, our answer is D.

Exponential Functions IV

Which of the following equations corresponds to the graph below?



A. $y = 2^x + 1$

B. $y = 2^{-x} + 1$

C. $y = 3^x + 1$

D. $y = 3^{-x} + 1$

E. $y = 4^x + 1$

Solution

Answer: C

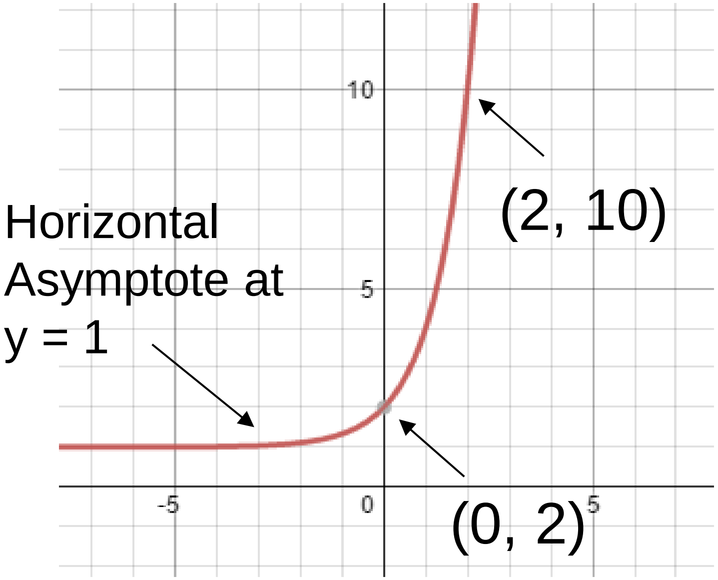
Justification: First, notice that we have applied transformations (**vertical translation**) to the exponential functions for creating our new functions. In our case, every exponential function is shifted vertically by $+1$ unit. As a result of the vertical shift, the horizontal asymptote has moved from $y = 0$ to $y = 1$. (1)

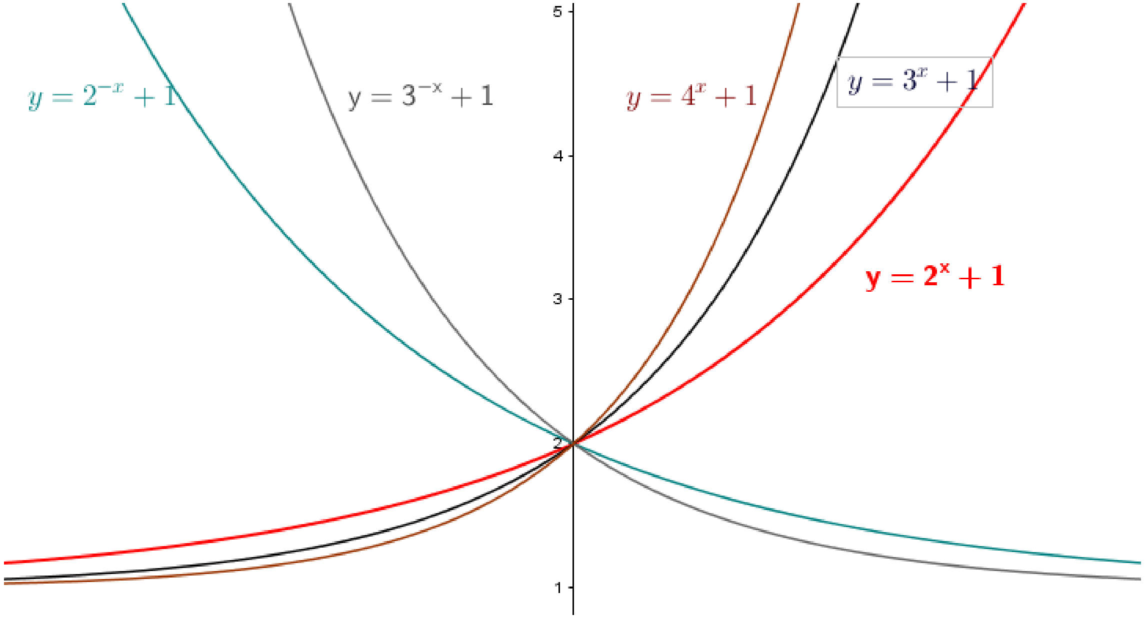
Second, our graph represents a function that is increasing. A **function is increasing** on an interval, if for any x_1 and x_2 in the interval then $x_1 < x_2$ implies $f(x_1) < f(x_2)$. Thus, B and D cannot be our answer since these two functions are decreasing functions. A **function is decreasing** on an interval, if for any x_1 and x_2 in the interval then $x_1 < x_2$ implies $f(x_1) > f(x_2)$. (2)

Solution Continued

Third, the y-intercept is $(0, 2)$ and another point on the graph is $(2, 10)$ (this point was chosen because it was easy to read the values off the graph). That is, when $x = 2$, $y = 3^2 + 1 = 10$. (3)

Consequently, the option that satisfies (1), (2), and (3) is **C**. Thus, our answer is C.





Exponential Functions V

Titanium 44 or Ti-44 is an important radioactive isotope that is produced in significant quantities during the core-collapse of supernovae. Ti-44 has a **half-life** of $60\ years$ and decays by electron capture. If you begin with a sample of N_0 quantity (measured in grams, moles, etc.) of Ti-44, what exponential function, $N(t)$, can be used to represent the radioactive decay of Ti-44 after some time t?

A. $N(t) = \frac{1}{2} N_0^{60\,t}$

B. $N(t) = \frac{1}{2} N_0^{(60/t)}$

C. $N(t) = \frac{1}{2} N_0^{(t/60)}$

D. $N(t) = N_0 \left(\frac{1}{2}\right)^{(60/t)}$

E. $N(t) = N_0 \left(\frac{1}{2}\right)^{(t/60)}$

Retrieved from http://www.ridehelix.ca/

Solution

Answer: E

Justification: Ti-44 has a half-life of 60 years and decays by electron capture. This means that after 60 years, a sample of Ti-44 will have lost one half of its original radioactivity.

In general, exponential decay processes can be described by $N(t) = N_0\, e^{-\lambda t}$ or $N(t) = N_0\, (\frac{1}{2})^{(t/t_{1/2})}$, where t is the time, $t_{1/2}$ is the half-life of the decaying quantity, $N(t)$ is the remaining quantity (not yet decayed) after time t, N_0 is the initial quantity (when $t = 0$) of the substance, and λ is a positive number called the decay constant.

Solution

Answer: E

Options **A** and **C**: $N(t) = \frac{1}{2} N_0{}^{60\,t}$ and $N(t) = \frac{1}{2} N_0{}^{(t/60)}$. When $t \to \infty, N(t) \to \infty$, which means that A and C describe exponential growths.

Option **B**: $N(t) = \frac{1}{2} N_0{}^{(60/t)}$. When $t \to \infty,\ \frac{60}{t} \to 0$, which means that $N_0{}^{(60/t)} \to N_0{}^{(0)} \to 1$. That is, $N(t) \to \frac{1}{2}$.

Option **D**: $N(t) = N_0 \left(\frac{1}{2}\right)^{(60/t)}$. When $t \to \infty,\ \frac{60}{t} \to 0$, which means that $\left(\frac{1}{2}\right)^{(60/t)} \to \left(\frac{1}{2}\right)^{(0)} \to 1$. That is, $N(t) \to N_0$.

Solution

Answer: E

Option **E**: $N(t) = N_0\left(\frac{1}{2}\right)^{(t/60)}$. When $t \to \infty, \frac{t}{60} \to \infty$, which means that $\left(\frac{1}{2}\right)^{(t/60)} \to \left(\frac{1}{2}\right)^{(\infty)} \to 0$. That is, $N(t) \to 0$.

Note that in options A, B, C, and D, $N(t)$ does not approach 0.

Remember, in an exponential decay, the remaining quantity, $N(t)$, of a substance approaches zero as t approaches infinity.

Thus, **E** is the correct answer.

Ti- 44: http://astro.triumf.ca/publications/categories/titanium-44

Half-life: https://en.wikipedia.org/wiki/Half-life

Exponential Functions VI

Customers of the Bank of Montreal (BMO) can open savings account to earn interest on their investments at an annual interest rate of 0.75%, compounded monthly. If your initial investment with BMO is P_0, what exponential function, $P(t)$, can be used to represent the future value of your investment? Let t be the number of years your investment is left in the bank.

A. $P(t) = 1.0075P_0^{12t}$

B. $P(t) = 1 + (0.0075P_0)^{12t}$

C. $P(t) = P_0(1.0075)^{12t}$

D. $P(t) = 1 + P_0(0.0075)^{12t}$

E. $P(t) = P_0^{12t} + 1.0075P_0$

http://steinmanfinancialnetwork.com/tax-free-savings-accounts/

Solution

Answer: C

0.75% = 0.0075

Justification: There are several ways to earn interest on the money you deposit in a bank. If the interest is calculated once a year, then the interest is called a **simple interest**. If the interest is calculated more than once a year, then it is called a **compound interest**.

In our case, it will be a 0.75% annual interest rate compounded monthly. That is, the interest will be compounded 12 times per year.

Options **A** and **E**: $P(t) = 1.0075\, {P_0}^{12t}$ and $P(t) = {P_0}^{12t} + \ 1.0075\, P_0$. These two options are too good to be true. Imagine if you were to invest $\$100$ with BMO, then by the end of the first year, you would have made more than a septillion dollars (more than 10^{24}).

Solution

Answer: C

Options **B** and **D**: $P(t) = 1 + (0.0075\,P_0)^{12t}$ and $P(t) = 1 + P_0(0.0075)^{12t}$. You will lose your investment with these two options. Imagine if you were to invest $\$100$ with BMO, then by the end of the first year and beyond, you would have lost $\$99$. In fact, over time (5, 10, or more years later), your investment would only be worth $\$1$.

Thus, **C** is the correct answer. Check the table below:

Year	P_0	A	B	C	D	E
1	\$100	$\$10^{24}$	\$1.03	**\$109.38**	\$1.00	$\$10^{24}$
2	\$100	$\$10^{48}$	\$1.00	**\$119.64**	\$1.00	$\$10^{48}$
3	\$100	$\$10^{72}$	\$1.00	**\$130.86**	\$1.00	$\$10^{72}$
4	\$100	$\$10^{96}$	\$1.00	**\$143.14**	\$1.00	$\$10^{96}$

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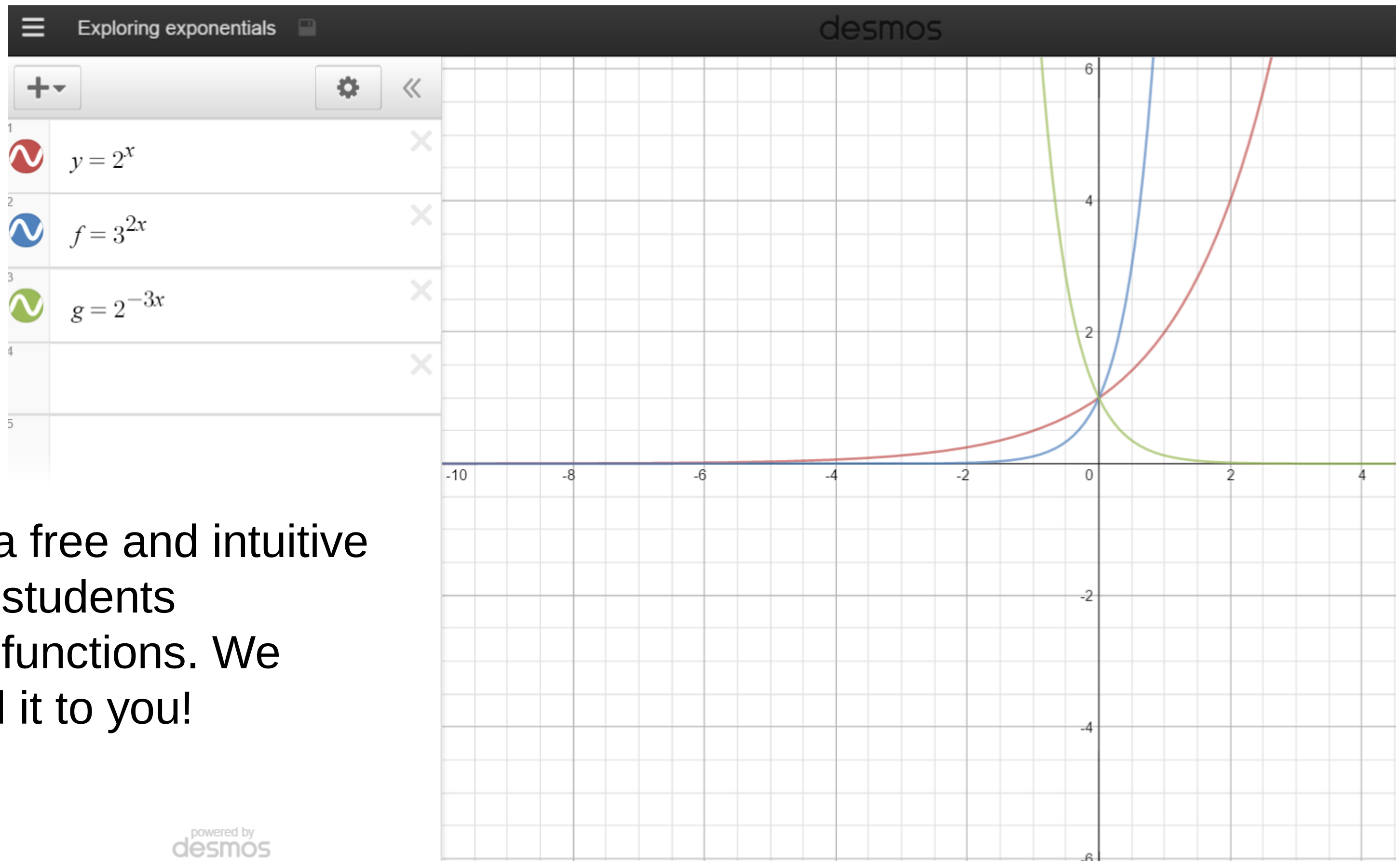

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